Polynomial Normal Forms with Exponentially Small Remainder for Analytic Vector Fields

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Abstract

To be done. Printed on January 23, 2006. *Version prélininaire pour fixer les notations*

Contents

1 Introduction

A key tool in the study of the dynamics of vector fields near an equilibrium point is the theory of normal forms, invented by Poincaré, which gives simple forms to which a vector field can be reduced close to the equilibrium [1],[6]. In the class of formal vector valued vector fields the problem can be easily solved [1], whereas in the class of analytic vector fields convergence issues of the power series giving the normalizing transformation generally occurs [6], [13]. Nevertheless the study of the dynamics in a neighborhood of the origin, can very often be carried out via a normalization up to finite order (see for instance [8], [11] **Reference à completer**). Normal forms are not unique and various characterization exist in the literature [7], **Reference à completer**. In this paper we will consider the version given in [7]:

Theorem 1.1 (Unperturbed NF-Theorem) *Let V be a smooth (resp. analytic) vector field defined on a neighborhood of the origin in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$.*

Then, for any integer $p \geq 2$, there are polynomials $\mathcal{Q}_p, \mathcal{N}_p : \mathbb{R}^m \to \mathbb{R}^m$ (resp. $\mathbb{C}^n \to \mathbb{C}^n$) , of degree $\leq p$, satisfying

$$\mathcal{Q}_p(0) = \mathcal{N}_p(0) = 0, D\mathcal{Q}_p(0) = D\mathcal{N}_p(0) = 0$$

such that under the near identity change of variable $X = Y + \mathcal{Q}_p(Y)$, the vector field

$$\frac{dX}{dt} = V(X) \tag{1}$$

becomes

$$\frac{dY}{dt} = LY + \mathcal{N}_p(Y) + \mathcal{R}_p(Y) \tag{2}$$

where $DV(0) = L$, where the remainder $\mathcal{R}_p$ is a smooth (resp. analytic) function satisfying $\mathcal{R}_p(X) = \mathcal{O}(\|X\|^{p+1})$ and where the normal form polynomial $\mathcal{N}_p$ of degree p is characterized by

$$\mathcal{N}_p(\mathrm{e}^{tL^*}Y) = \mathrm{e}^{tL^*}\mathcal{N}_p(Y)$$

for all $Y \in \mathbb{R}^m$ (resp. in $\mathbb{C}^m$) and $t \in \mathbb{R}$ or equivalently

$$D\mathcal{N}_p(Y)L^*Y = L^*\mathcal{N}_p(Y)$$

where L^* is the adjoint of L. Moreover, if T is a unitary linear map which commutes with V then for every Y,

$$\mathcal{Q}_p(TY) = T\mathcal{Q}_p(Y) \qquad \mathcal{N}_p(TY) = T\mathcal{N}_p(Y).$$

Similarly, if V is reversible with respect to some linear symmetry S ($S^2 = \mathrm{I_d}$), i.e. if v anticommutes with this symmetry, then for every Y,

$$\mathcal{Q}_p(SY) = S\mathcal{Q}_p(Y) \qquad \mathcal{N}_p(SY) = -S\mathcal{N}_p(Y).$$

This version of the Normal Form Theorem up to finite order has two advantages : firstly, it works for a non semi-simple linear operator L and secondly the characterization of the normal form $\mathcal{N}_p$ involves the adjoint L^* of the operator L and not simply the diagonalizable part of L. This leads to simpler normal forms.

Since the usual way to study the dynamics of vector fields close to an equilibrium is to see the full vector field as a perturbation of its normal form $L+\mathcal{N}_p$ by higher order terms, it happens to be of great interest to obtain sharp upper bounds of the remainders $\mathcal{R}_p$. A similar theory of resonant normal forms was developed for Hamiltonians systems written in action-angle coordinates (see for instance [2], [4], [12]). A sticking result obtained by Nekhoroshev [9], [10], in order to study the stability of the action variables over exponentially large interval of time, is that up to an optimal choice of the order p of the normal form , the remainder can be made exponentially small. For more details of such Normal Form Theorems with exponentially small remainder we refer to [12]. A similar result of exponential smallness of the remainder was also obtained by Giorgilli and Posilicano in [5] for a reversible system with a linear part composed of Harmonic oscillators.

The aim of the present paper is to prove that such a result of exponential smallness of the remainder is still true for any analytic vector fields provided that the spectrum of its linearization satisfies some "nonresonance assumptions" which enable to control the small divisor effects. For a subset $\mathcal{Z}$ of $\mathbb{Z}^m$, for $K \in \mathbb{N}$, and for $\gamma > 0$, a vector

$\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^m$, is said to be $\boldsymbol{\gamma}, K$*-nonresonant modulo* $\mathcal{Z}$ if for every $k \in \mathbb{Z}^m$ with $|k| \leq K$,

$$|\langle \lambda, k \rangle| \geq \gamma \qquad \text{when} \qquad k \notin \mathcal{Z}.$$

Similarly, for $\gamma > 0, \tau > m-1$, λ is said to be $\boldsymbol{\gamma}, \boldsymbol{\tau}$*-Diophantine modulo* $\mathcal{Z}$ if for every $k \in \mathbb{Z}^m$,

$$|\langle \lambda, k \rangle| \geq \frac{\gamma}{|k|^\tau} \qquad \text{when} \qquad k \notin \mathcal{Z}.$$

where for $k = (k_1, \cdots, k_m) \in \mathbb{Z}^m$, $|k| := |k_1| + \cdots + |k_m|$. In the problem of normal forms, the small divisors appears as eigenvalues of the homological operator giving the normal forms by induction (see Subsection 2.1 and Lemma 2.5). To control these small divisors let us introduce the following definitions :

Definition 1.2 *Let us define* $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^m$*,* $K \in \mathbb{N}$*,* $\gamma > 0$ *and* $\tau > m-1$*.*

(a) *The vector* λ *is said to be* $\boldsymbol{\gamma}, K$**-homologically nonresonant** *if for every* $\alpha \in \mathbb{N}^m$ *with* $2 \leq |\alpha| \leq K$*, and every* $j \in \mathbb{N}$*,* $1 \leq j \leq m$*,*

$$|\langle \lambda, \alpha \rangle - \lambda_j| \geq \gamma \qquad \textit{when} \qquad \langle \lambda, \alpha \rangle - \lambda_j \neq 0.$$

(b) *The vector* λ *is said to be* $\boldsymbol{\gamma}, \boldsymbol{\tau}$**-homologically Diophantine** *if for every* $\alpha \in \mathbb{N}^m$*,* $|\alpha| \geq 2$*,*

$$|\langle \lambda, \alpha \rangle - \lambda_j| \geq \frac{\gamma}{|\alpha|^\tau} \qquad \textit{when} \qquad \langle \lambda, \alpha \rangle - \lambda_j \neq 0.$$

(c) *For a linear operator* L *in* $\mathbb{R}^m$*, let us denote* $\lambda_1, \cdots, \lambda_m$ *its eigenvalues and* $\lambda_L := (\lambda_1, \cdots, \lambda_m)$*. Then* L *is said to be* $\boldsymbol{\gamma}, K$**-homologically nonresonant** *(resp.* $\boldsymbol{\gamma}, \boldsymbol{\tau}$**-homologically Diophantine***) if* λ_L *is so.*

Remark 1.3 Observe that in the above definitions, the components of α are a nonnegative integers.

In what follows we use Arnold's notations [1] for denoting matrices under complex Jordan normal forms : $\boldsymbol{\lambda}^{\mathbf{2}}$ denotes the 2×2 complex Jordan block corresponding to $\lambda \in \mathbb{C}$ whereas $\boldsymbol{\lambda}.\boldsymbol{\lambda}$ represents 2×2 complex diagonal matrix $\mathrm{diag}(\lambda, \lambda)$, i.e.

$$\boldsymbol{\lambda}^{\mathbf{2}} := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{whereas} \qquad \boldsymbol{\lambda}.\boldsymbol{\lambda} := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

A matrix under complex Jordan normal form is then denoted by the products of the name of its Jordan blocs. Moreover since for real matrices the Jordan blocks corresponding to non zero matrices occur by pairs $\boldsymbol{\lambda}^r$ and $\overline{\boldsymbol{\lambda}}^r$ we shorten their name as follows : for $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, $\mathbf{0^2}.\boldsymbol{\lambda}_{\mathbf{1}}^{\boldsymbol{r_1}}.\boldsymbol{\lambda}_{\mathbf{2}}^{\boldsymbol{r_2}}.\overline{\boldsymbol{\lambda}}_{\mathbf{1}}^{\boldsymbol{r_1}}.\overline{\boldsymbol{\lambda}}_{\mathbf{2}}^{\boldsymbol{r_2}}$ is simply denoted by $\mathbf{0^2}.\boldsymbol{\lambda}_{\mathbf{1}}^{\boldsymbol{r_1}}.\boldsymbol{\lambda}_{\mathbf{2}}^{\boldsymbol{r_2}}|_{\mathbb{C}}$. Moreover, when one works with vector fields in $\mathbb{R}^m$, one may want to remain in $\mathbb{R}^m$ and thus to use real Jordan normal forms for the linearization of the vector field. So, for $\mu \in \mathbb{R}$ and $\lambda = x + \mathrm{i}y \in \mathbb{C} \setminus \mathbb{R}$, we denote by $\mu^2\lambda^2|_{\mathbb{R}}$ the real Jordan matrix

$$\begin{pmatrix} \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} x & -y \\ y & x \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{pmatrix}.$$

Finally, we equip $\mathbb{R}^m$ and $\mathbb{C}^m$ with the canonical inner product and norm, i.e. for $X = (X_1, \cdots, X_m) \in \mathbb{C}^m$, $\quad \|X\|^2 := \langle X, X\rangle = \sum\limits_{j=1}^{m} \overline{X_j} X_j$. We are now ready to state our main result:

Theorem 1.4 (Unperturbed NF-Theorem with exponentially small remainder) *Let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e.*

$$V(X) = LX + \sum_{k\geq 2} V_k[X^{(k)}] \tag{3}$$

where L is a linear operator in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) and where V_k is bounded k-linear symmetric and

$$\|V_k[X_1, \cdots, X_k]\| \leq c\frac{\|X_1\| \cdots \|X_k\|}{\rho^k} \tag{4}$$

with $c, \rho > 0$ independent of k.

(a) *If L is semi-simple and under real (resp. complex) Jordan normal forms, then*

   (i) *if L is γ, τ-homologically Diophantine, then for every $\delta > 0$ such that $p_{\rm opt} \geq 2$, the remainder $\mathcal{R}_p$, given by the Normal Form Theorem 1.1 for $p = p_{\rm opt}$, satisfies*

$$\sup_{\|Y\|\leq\delta} \|\mathcal{R}_{p_{\rm opt}}(Y)\| \leq M_\tau \delta^2 \mathrm{e}^{-\frac{w}{\delta^b}} \tag{5}$$

   with

$$b = \frac{1}{1+\tau}, \quad p_{\rm opt} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right], \quad w = \frac{1}{\mathrm{e}C^b}$$

   and

$$M_\tau = \frac{10}{9} cC^2 \left\{ \left(\mathfrak{m}\sqrt{\frac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau} \right\}$$

   where

$$C = \frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m + 2\right) ac + 3\rho\right\}, \qquad \mathfrak{m} = \sup_{p\in\mathbb{N}} \frac{\mathrm{e}^2\, p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}, \qquad a = \gamma^{-1}.$$

   (ii) *if L is γ, K-homologically nonresonant, then for every $\delta > 0$ such that $K \geq p_{\rm opt} \geq 2$ then the remainder $\mathcal{R}_p$ given by the Normal Form Theorem 1.1 for $p = p_{\rm opt}$ satisfies (5) with $\tau = 0$, i.e. $b = 1$.*

(b) *If L is not semi simple, then*

   (i) *For $L = \mathbf{0}^2$, $L = \mathbf{0}^3$, $L = \mathbf{0}^2.\mathbf{i}\boldsymbol{\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ and $L = (\mathbf{i}\boldsymbol{\omega})^2|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ estimate (5) still holds with $\tau = 0$, i.e $b = 1$ and respectively with $a = 1$, $a = 1$, $a = \max(1, \omega^{-1})$ and $a = \max(1, \omega^{-1})$.*

   (ii) *For $L = \mathbf{0}^2.\mathbf{i}\boldsymbol{\omega_1}.\cdots.\mathbf{i}\boldsymbol{\omega_q}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ and $L = (\mathbf{i}\boldsymbol{\omega_1})^2.\mathbf{i}\boldsymbol{\omega_2}.\cdots.\mathbf{i}\boldsymbol{\omega_q}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$, where $\omega := (\omega_1, \cdots, \omega_q, -\omega_1, \cdots, -\omega_q) \in \mathbb{R}^{2q}$ is γ, τ-homologically diophantine, estimate (5) still holds with $a = \max(2^{-\tau}, \gamma^{-1})$.*

Remark 1.5 Stirling's formula ensures that $\mathfrak{m}$ is finite.

Remark 1.6 A semi simple matrix under complex Jordan normal form is simply a diagonal matrix whereas a real semi simple matrix under real Jordan normal form is the direct sum of a diagonal matrix with 2×2 blocks of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ with $x, y \in \mathbb{R}$.

Remark 1.7 The characterization of the normal form and the exponentially small estimates are invariant under unitary changes of coordinates. Indeed, if we perform in (2) a unitary change of coordinates $Y = Q\widetilde{Y}$ where Q is a unitary linear operator ($Q^* = Q^{-1}$), then it becomes

$$\frac{d\widetilde{Y}}{dt} = \widetilde{L}\widetilde{Y} + \widetilde{\mathcal{N}}_p(\widetilde{Y}) + \widetilde{\mathcal{R}}_p(\widetilde{Y})$$

with $\widetilde{L} = Q^{-1}LQ$, $\widetilde{\mathcal{N}_p}(\widetilde{Y}) = Q^{-1}\mathcal{N}_p(Q\widetilde{Y})$, $\widetilde{\mathcal{R}}_p(Y) = Q^{-1}\mathcal{R}_p(QY)$, where $\widetilde{\mathcal{N}_p}$ satisfies the same normal form criteria as $\mathcal{N}_p$, i.e. $\widetilde{\mathcal{N}}(\mathrm{e}^{t\widetilde{L}^*}\widetilde{Y}) = \mathrm{e}^{t\widetilde{L}^*}\widetilde{\mathcal{N}}(\widetilde{Y})$ and where $\widetilde{\mathcal{R}}_p$ admits the same exponentially small upper bound as $\mathcal{R}_p$ given by (5).

However, when Q is not unitary then $\widetilde{\mathcal{N}_p}$ satisfies a slightly different normal form criteria given by

$$\widetilde{\mathcal{N}}(\mathrm{e}^{t\check{L}}\widetilde{Y}) = \mathrm{e}^{t\check{L}}\widetilde{\mathcal{N}}(\widetilde{Y})$$

where $\check{L} = Q^{-1}L^*Q$ which is not equal to $\widetilde{L}^*$ when Q is not unitary. In this case, $\widetilde{\mathcal{R}}_{p_{\mathrm{opt}}}$ also admits a slightly different upper bound given by

$$\sup_{\|\widetilde{Y}\| \le \widetilde{\delta}} \|\widetilde{\mathcal{R}}_{p_{\mathrm{opt}}}(Y)\| \le M_\tau |||Q^{-1}|||\ |||Q|||^2\ \widetilde{\delta}^2\ \mathrm{e}^{-\frac{w}{|||Q|||^b\ \widetilde{\delta}^b}}.$$

The above remark enables to state a corollary without assuming that L is under real or complex Jordan normal form

Corollary 1.8 *Let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e. satisfying (3) and (4). Denote $L = DV(0)$ and let Q be an invertible matrix such that $J = QLQ^{-1}$ is under real (resp. complex) Jordan normal form.*

Then, there are polynomials $\mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}, \mathsf{N}_{\mathsf{p}_{\mathrm{opt}}} : \mathbb{R}^m \to \mathbb{R}^m$ (resp. $\mathbb{C}^n \to \mathbb{C}^n$) , of degree $\le p_{\mathrm{opt}}$, satisfying $\mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}(0) = \mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(0) = 0, D\mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}(0) = D\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(0) = 0$ such that under the near identity change of variable $X = Y + \mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}(Y)$, the vector field (1) becomes

$$\frac{dY}{dt} = LY + \mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(Y) + \mathsf{R}_{\mathsf{p}_{\mathrm{opt}}}(Y)$$

where the remainder $\mathsf{R}_{\mathsf{p}_{\mathrm{opt}}} = \mathcal{O}(\|Y\|^{\mathsf{p}_{\mathrm{opt}}+1})$ is analytic and where $\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}$ satisfies the normal form criteria

$$\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(\mathrm{e}^{t\check{L}}Y) = \mathrm{e}^{t\check{L}}\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(Y) \qquad \text{with } \check{L} = Q^{-1}J^*Q$$

for all $Y \in \mathbb{R}^m$ (resp. in $\mathbb{C}^m$) and $t \in \mathbb{R}$. Moreover,

(a) *if L is semi-simple and γ, τ-homologically Diophantine, then for every $\delta > 0$ such that $\mathsf{p}_{\mathrm{opt}} \ge 2$, the remainder $\mathsf{R}_{\mathsf{p}_{\mathrm{opt}}}$ satisfies*

$$\sup_{\|Y\| \le \delta} \|\mathsf{R}_{\mathsf{p}_{\mathrm{opt}}}(Y)\| \le \mathsf{M}_\tau \delta^2 \mathrm{e}^{-\frac{\mathsf{w}}{\delta^b}} \tag{6}$$

with

$$b = \frac{1}{1+\tau}, \quad \mathsf{p}_{\mathrm{opt}} = \left[\frac{1}{\mathrm{e}(\mathsf{C}\delta)^b}\right], \quad \mathsf{w} = \frac{1}{\mathrm{e}\mathsf{C}^b}$$

and

$$\mathsf{M}_\tau = \frac{10}{9} c |||Q||| \, |||Q^{-1}||| \; \mathsf{C}^2 \left\{ \left(\mathfrak{m}\sqrt{\frac{27}{8\mathrm{e}}} \right)^{1+\tau} + (2\mathrm{e})^{2+2\tau} \right\}$$

where

$$C = \frac{\sqrt{m}}{\rho^2} \; \left\{ \left(\frac{5}{2}m + 2\right) ac|||Q|||^2 \, |||Q^{-1}|||^2 + 3\rho |||Q||| \, |||Q^{-1}||| \right\}, \quad \mathfrak{m} = \sup_{p\in\mathbb{N}} \frac{\mathrm{e}^2 \; p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$$

and $a = \gamma^{-1}$*;*

(b) *if* L *is* γ, K*-homologically nonresonant, then for every* $\delta > 0$ *such that* $K \geq \mathsf{p}_{\mathrm{opt}} \geq 2$ *then the remainder* $\mathsf{R}_{\mathsf{p}_{\mathrm{opt}}}$ *satisfies (6) with* $\tau = 0$*, i.e.* $b = 1$*;*

(c) *for* $J = \mathbf{0}^2$*,* $J = \mathbf{0}^3$*,* $J = \mathbf{0}^2.\mathbf{i}\boldsymbol{\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ *and* $J = (\mathbf{i}\boldsymbol{\omega})^2|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ *estimate (6) still holds with* $\tau = 0$*, i.e* $b = 1$ *and respectively with* $a = 1$*,* $a = 1$*,* $a = \max(1, \omega^{-1})$ *and* $a = \max(1, \omega^{-1})$*;*

(d) *for* $J = \mathbf{0}^2.\mathbf{i}\boldsymbol{\omega_1}.\cdots.\mathbf{i}\boldsymbol{\omega_q}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ *and* $J = (\mathbf{i}\boldsymbol{\omega_1})^{\mathbf{2}}.\mathbf{i}\boldsymbol{\omega_2}.\cdots.\mathbf{i}\boldsymbol{\omega_q}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$*, where* $\omega := (\omega_1, \cdots, \omega_q, -\omega_1, \cdots, -\omega_q) \in \mathbb{R}^{2q}$ *is* γ, τ*-homologically diophantine, estimate (6) still holds with* $a = \max(2^{-\tau}, \gamma^{-1})$*.*

Proof. Starting with (1), perform a first change of coordinates $X = Q^{-1}\widetilde{X}$ to obtain a vector field $\widetilde{V}$ such that $D\widetilde{V}(0) = J$ is under Jordan normal form, then apply Theorem 1.4, i.e perform a second change of coordinates $\widetilde{X} = \widetilde{\mathcal{Q}}_{\mathsf{p}_{\mathrm{opt}}}(\widetilde{Y})$ and finally perform a last change of coordinates $\widetilde{Y} = QY$ to get the desired result. □

Reste à commenter ce théorème puis à énoncer les résultats pour les champs perturbés.

2 Exponential estimates for unperturbed vector fields

This section is devoted to the proof of Theorem 1.4. We first recall in few words the proof of Theorem 1.1.

2.1 Normalization and Homological equations

Let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e. a vector field satisfying (3) and (4). Let $\mathcal{H}$ be the space of the polynomial $\Phi : \mathbb{R}^m \mapsto \mathbb{R}^m$ (resp. $\mathbb{C}^m \mapsto \mathbb{C}^m$) and let $\mathcal{H}_k$ be the space of the homogeneous ones of degree k. We are interested in polynomial changes of variables, of the form $X = Y + \mathcal{Q}_p(Y)$ with

$$\mathcal{Q}_p(Y) = \sum_{2\leq k\leq p} \Phi_k(Y), \quad \Phi_k \in \mathcal{H}_k$$

such that by the change of variable, equation (1) becomes of the form (2) with

$$\mathcal{N}_p(Y) = \sum_{2\leq k\leq p} N_k(Y), \qquad N_k \in \mathcal{H}_k,$$

where $\mathcal{N}_p$ is as simple as possible. A basic identification of powers of Y leads to

$$\begin{aligned} &\{1 + \sum_{2\leq k\leq p} D\Phi_k(Y)\}\{LY + \sum_{2\leq k\leq p} N_k(Y) + \mathcal{R}_p(Y)\} \\ &= L\{\sum_{1\leq k\leq p} \Phi_k(Y)\} + \sum_{q\geq 2} V_q \left[\{\sum_{1\leq k\leq p} \Phi_k(Y)\}^{(q)}\right]. \end{aligned} \tag{7}$$

where $\Phi_1(Y) = Y$. This leads to the following hierarchy of homological equations in $\mathcal{H}_n$ for $2 \leq n \leq p$,

$$\mathcal{A}_L \Phi_n + N_n = F_n, \tag{E$_n$}$$

with

$$F_n = -\sum_{2\leq k\leq n-1} D\Phi_k.N_{n-k+1} + \sum_{2\leq q\leq n} \sum_{p_1+\cdots+p_q=n} V_q[\Phi_{p_1}, \cdots, \Phi_{p_q}], \tag{8}$$

where some sums are empty and where $\mathcal{A}_L$ is the homological operator given by

$$(\mathcal{A}_L\Phi)(Y) = D\Phi(Y).LY - L\Phi(Y).$$

Observe that $\mathcal{A}$ induces on each $\mathcal{H}_n$ a linear endomorphism denoted by $\mathcal{A}_L|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n$. Generally $\mathcal{A}_L|_{\mathcal{H}_n}$ is not invertible. So when F_n lies in the range $\mathrm{ran}(\mathcal{A}_L|_{\mathcal{H}_n})$ of $\mathcal{A}_L|_{\mathcal{H}_n}$ one can take $N_n = 0$ and for Φ_n any preimage of F_n whereas when F_n does not lie in $\mathrm{ran}(\mathcal{A}_L|_{\mathcal{H}_n})$, then one has to chose N_n in an appropriate supplementary space of $\mathrm{ran}(\mathcal{A}_L|_{\mathcal{H}_n})$ so that $F_n - N_n$ belongs to $\mathrm{ran}(\mathcal{A}_L|_{\mathcal{H}_n})$.

The key idea of the proof of Theorem 1.1 contained in [7] is to introduce an appropriate inner product on $\mathcal{H}$ such that the adjoint $\mathcal{A}_L^*$ of $\mathcal{A}_L$ is given by $\mathcal{A}_{L^*}$. Hence,

$$\mathcal{H}_n = \ker \mathcal{A}_L|_{\mathcal{H}_n} \overset{\perp}{\oplus} \mathrm{ran}\mathcal{A}_{L^*}|_{\mathcal{H}_n}, \qquad \mathcal{H}_n = \mathrm{ran}\mathcal{A}_L|_{\mathcal{H}_n} \overset{\perp}{\oplus} \ker \mathcal{A}_{L^*}|_{\mathcal{H}_n}.$$

Then for solving (E$_n$), we use the orthogonal projection π_n on $\ker \mathcal{A}_{L^*}|_{\mathcal{H}_n}$ for obtaining N_n and the pseudo-inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_n}^{-1}$ of $\mathcal{A}_L|_{\mathcal{H}_n}$ defined in $(\ker \mathcal{A}_{L^*})^\perp = \mathrm{ran}\mathcal{A}_L|_{\mathcal{H}_n}$ taking values in $(\ker \mathcal{A}_L|_{\mathcal{H}_n})^\perp$ for N_n, i.e.

$$N_n = \pi_n(F_n) \qquad \text{and} \qquad \Phi_n = \widetilde{\mathcal{A}_L}|_{\mathcal{H}_n}^{-1}((\mathrm{I_d} - \pi_n)(F_n)). \tag{9}$$

This completes the proof of theorem 1.1 and ensures that N_n belongs to $\ker \mathcal{A}_{L^*}|_{\mathcal{H}_n}$ and thus that $\mathcal{N}_p$ lies in $\ker \mathcal{A}_{L^*} := \{\mathcal{N}/D\mathcal{N}(Y)L^*Y - L^*\mathcal{N}(Y) = 0\}$.

To conclude this subsection, the appropriate inner product in $\mathcal{H}$ introduced in [7] is given by

$$\langle \Phi|\Phi'\rangle_{\mathcal{H}} = \sum_{j=1}^{m} \left\langle \Phi_j|\Phi'_j \right\rangle$$

with $\Phi = (\Phi_1, \cdots, \Phi_m)$, $\Phi' = (\Phi'_1, \cdots, \Phi'_m)$, where for any pair of polynomial $P, P' : \mathbb{R}^m \to \mathbb{R}$ (resp. $\mathbb{C}^m \to \mathbb{C}$) ,

$$\langle P|P'\rangle = \overline{P}(\partial_Y)P'(Y)|_{Y=0}.$$

where by definition $\overline{P}(Y) := \overline{P(\overline{Y})}$. E.g, for all positive integers $\alpha_1, \cdots, \alpha_m$, $\beta_1, \cdots, \beta_m$

$$\left\langle Y^{\alpha_1}.\cdots.Y^{\alpha_m}|Y^{\beta_1}.\cdots.Y^{\beta_m}\right\rangle = \alpha_1!.\cdots.\alpha_m!\ \delta_{\alpha_1,\beta_1}.\cdots.\delta_{\alpha_m,\beta_m}$$

where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$, and 0 otherwise. It what follows we norm $\mathcal{H}_n$ with the associated euclidian norm $|\Phi|_2 := \sqrt{\langle \Phi|\Phi\rangle_{\mathcal{H}}}$

2.2 Exponential upper bounds for the remainder: main results

Main result. We want to give an estimate on $\mathcal{R}_p(Y)$ depending on p and on the size of the ball where Y lies. Given the size of this ball, the aim is to optimize the degree p of the normal form, and show that $\mathcal{R}_p(Y)$ can be made exponentially small with respect to δ. For unperturbed vector fields, all follows from the following proposition which ensures that the exponentially estimates of the remainder follows from the estimates of the growth with respect to k of the euclidian norm of the pseudo inverse of $\mathcal{A}_L|_{\mathcal{H}_k}$.

Remark 2.1 A priori the pseudo inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ is only defined from $(\ker \mathcal{A}_{L^*})^{\perp} = \mathrm{ran}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_{\mathrm{k}}}$ onto $(\ker \mathcal{A}_L|_{\mathcal{H}_k})^{\perp}$. From now on, we extend it on the whole space $\mathcal{H}_k$ as follows

$$\mathcal{A}_L|_{\mathcal{H}_k}\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\Phi = \Phi \quad \text{for } \Phi \in (\ker \mathcal{A}_{L^*})^{\perp}, \qquad \widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\Phi = 0, \quad \text{for } \Phi \in \ker \mathcal{A}_{L^*}.$$

Proposition 2.2 (Exponential estimates of the remainder) *Let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e. a vector field satisfying (3) and (4). Denote*

$$a_k := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 = \sup_{|\Phi|_2=1} \left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}.\Phi\right|_2.$$

Then, if there exits $K \geq 2$, $a \geq 0$ and $\tau \geq 0$ such that $a_k \leq ak^{\tau}$ for every k with $2 \leq k \leq K \leq +\infty$, then for every $\delta > 0$ such that $K \geq p_{\rm opt} \geq 2$ the remainder $\mathcal{R}_p$ given by the Normal Form Theorem 1.1 for $p = p_{\rm opt}$ satisfies

$$\sup_{\|Y\|\leq\delta} \|\mathcal{R}_{p_{\rm opt}}(Y)\| \leq M\ \delta^2 \mathrm{e}^{-\frac{w}{\delta^b}}$$

with

$$b = \frac{1}{1+\tau}, \quad p_{\rm opt} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right], \quad w = \frac{1}{\mathrm{e}C^b}, \quad M = \frac{10}{9}cC^2\left\{\left(\mathfrak{m}\sqrt{\frac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau}\right\}$$

where $C = \frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m+2\right)ac+3\rho\right\}$, $\mathfrak{m} = \sup_{p\in\mathbb{N}} \dfrac{\mathrm{e}^2\, p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$ *and where for a real number* x*, we denote by* $[x]$ *its entire part.*

Remark 2.3 Stirling's formula ensures that $\mathfrak{m}$ is finite.

The proof of this proposition is performed in two main steps. We first prove that roughly speaking, R_p admits an upper bound of the form

$$\sup_{\|Y\|\leq\delta} \|\mathcal{R}_p(Y)\| \leq M(C\delta)^{p+1}(p!)^{1+\tau}.$$

where M depends on τ but not on δ and p. Then we optimize p (see Lemma 2.17), so that $(C\delta)^{p+1}(p!)^{\tau+1}$ is exponentially small for $p = p_{\mathrm{opt}}$. In fact, the upper bound for $\mathcal{R}_p$ is a little bit more complicated (see Lemma 2.15) and we obtain it only for $(C\delta)^{\frac{1}{1+\tau}}p \leq \mathrm{e}^{-1}$, which is just enough to obtain the desired exponentially small upper bound of the remainder. The detailed proof of this proposition is postponed to subsection 2.3.

Remark 2.4 The euclidian norms a_k of the homological operator are invariant under unitary changes of coordinates. Indeed, if Q is a unitary linear operator, let us denote $L' = Q^{-1}LQ$ and $a'_k = |||\widetilde{\mathcal{A}_{L'}|_{\mathcal{H}_k}^{-1}}|||_2$. Then, since $\mathcal{A}_{L'}|_{\mathcal{H}_k} = \mathcal{T}_Q\mathcal{A}_L|_{\mathcal{H}_k}\mathcal{T}_Q^{-1}$ where $(\mathcal{T}_Q\Phi)(Y) = Q^{-1}\Phi(QY)$ and since $\mathcal{T}_Q$ is unitary when Q is unitary (see Appendix A.3), we get that $a'_k = a_k$ for every $k \geq 1$.

The semi-simple case. Theorem 1.4-(a) directly follows from proposition 2.2 and from the following lemma

Lemma 2.5 *Let* L *be a* $m\times m$ *matrix.*

(a) *Denote by* $\sigma(L) := \{\lambda_1, \cdots, \lambda_m\}$ *the spectrum of* L*. Then, for every* $k \geq 2$ *the spectrum* $\sigma(\mathcal{A}_L|_{\mathcal{H}_k})$ *of* $\mathcal{A}_L|_{\mathcal{H}_k}$ *is given by*

$$\sigma(\mathcal{A}_L|_{\mathcal{H}_k}) := \{\Lambda_{j,\alpha} := \langle\lambda_L, \alpha\rangle - \lambda_j, \quad 1\leq j\leq m,\ \alpha\in\mathbb{N}^m, |\alpha| = k\}. \tag{10}$$

(b) *If* L *is semi-simple and is under real or complex Jordan normal form, then for every* $k \geq 2$*,*

$$a_k := |||\widetilde{\mathcal{A}_L|_{\mathcal{H}_k}^{-1}}|||_2 \leq \max_{\substack{1\leq j\leq m, |\alpha|=k\\ \Lambda_{j,\alpha}\neq 0}} |\Lambda_{j,\alpha}|^{-1}.$$

Remark 2.6 When L is semi simple, under Jordan normal form, and γ, K-homologically nonresonant, the above lemma ensures that $a_k \leq \gamma^{-1}$ for $2 \leq k \leq K$ and if L is γ,τ-homologically Diophantine, then $a_k \leq \gamma^{-1}\ k^\tau$ for $k \geq 2$.

Proof of Lemma 2.5. **(a)**:Although this result is classical (see [3]), we give its short proof for self-contentness of the paper. Let Q be an invertible matrix such that $J = Q^{-1}LQ$ is under complex Jordan normal form and observe that $\mathcal{A}_L|_{\mathcal{H}_k} = \mathcal{T}_Q^{-1}\mathcal{A}_J|_{\mathcal{H}_k}\mathcal{T}_Q$ where $(\mathcal{T}_Q\Phi)(Y) = Q^{-1}\Phi(QY)$. Hence the spectrum of $\mathcal{A}_L|_{\mathcal{H}_k}$ is equal to the spectrum of $\mathcal{A}_J|_{\mathcal{H}_k}$. Let $\{c_j\}_{1\leq j\leq m}$ be the canonical basis of $\mathbb{C}^m$. Then, since J is under Jordan normal form, we have $Jc_j = \lambda_j c_j + \delta_{j-1} c_{j-1}$ with $\delta_0 = 0$ and where $\delta_{j-1} = 0$ if $\lambda_j \neq \lambda_{j-1}$ and $\delta_{j-1} = 0$ or 1 otherwise. Let $\{P_{j,\alpha}\}_{1\leq j\leq m, \alpha\in\mathbb{N}^m, |\alpha|=k}$ be the basis of $\mathcal{H}_k$ given by

$$P_{j,\alpha}(Y) := Y_1^{\alpha_1}.\cdots.Y_m^{\alpha_m} c_j$$

we order this basis with the lexicographical order, i.e. $P_{j,\alpha} < P_{\ell,\beta}$ if the first non zero integer $\ell - j, \beta_1 - \alpha_1.\cdots.\beta_m - \alpha_m$ is positive. Within this order, $\mathcal{A}_J$ is upper triangular and

$$\mathcal{A}_J P_{j,\alpha} = (\langle \lambda_L, \alpha\rangle - \lambda_j) P_{j,\alpha} + \sum_{\ell=1}^{m} \alpha_\ell \delta_\ell P_{j,\alpha-\sigma_\ell+\sigma_{\ell+1}} - \delta_{j-1} P_{j-1,\alpha} \tag{11}$$

with $\sigma_\ell = (0,\cdots,0,1,\cdots,0)$ where the coefficient 1 is at the ℓ-th position. Hence the spectrum of $\mathcal{A}_J|_{\mathcal{H}_k}$ and thus the spectrum of $\mathcal{A}_L|_{\mathcal{H}_k}$ is given by (10).

(b) : We proceed in two steps.
Step 1. First assume that L is semi-simple and is under complex Jordan normal form i.e. assume that $L = J$ is diagonal. Then $\delta_j = 0$ for $1 \leq j \leq m$. Thus, by (11), $\mathcal{A}_L|_{\mathcal{H}_k}$ is also semi simple and $\{P_{j,\alpha}\}_{1\leq j\leq m, \alpha\in\mathbb{N}^m, |\alpha|=k}$ is a basis of eigenvectors of $\mathcal{A}_L|_{\mathcal{H}_k}$. For $\Phi \in \mathcal{H}_k$, let us denote

$$\Phi = \widehat{\Phi} + \overset{\vee}{\Phi}, \qquad \overset{\vee}{\Phi} = \pi_k \Phi \in \ker(\mathcal{A}_{L^*}|_{\mathcal{H}_k}), \qquad \widehat{\Phi} = \sum_{\substack{1\leq j\leq m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} \widehat{\Phi}_{j,\alpha} P_{j,\alpha} \in \operatorname{ran}(\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_{\mathrm{k}}}),$$

and $\mathsf{M} = \max\limits_{\substack{1\leq j\leq m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} |\Lambda_{j,\alpha}|^{-1}$. Then since $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1} \overset{\vee}{\Phi} = 0$ and $\langle P_{j,\alpha} | P_{\ell,\beta}\rangle_{\mathcal{H}} = 0$ for $(j,\alpha) \neq (\ell,\beta)$ we have

$$\begin{aligned}
\left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1} \Phi\right|_2^2 &= \sum_{\substack{1\leq j\leq m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} |\Lambda_{j,\alpha}|^{-2} \; |\widehat{\Phi}_{j,\alpha}|^2 \; |P_{j,\alpha}|_2^2 \\
&\leq \mathsf{M}^2 \sum_{\substack{1\leq j\leq m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} |\widehat{\Phi}_{j,\alpha}|^2 \; |P_{j,\alpha}|_2^2 \\
&= \mathsf{M}^2 \left|\widehat{\Phi}\right|_2^2
\end{aligned}$$

Finally, since $\langle \overset{\vee}{\Phi} | \widehat{\Phi}\rangle_{\mathcal{H}} = 0$,

$$\left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1} \Phi\right|_2 \leq \mathsf{M} \left|\Phi\right|_2. \tag{12}$$

Step 2. if L is real semi simple and is under real Jordan normal form then it is conjugated to to its complex Jordan normal form by a unitary matrix since

$$\begin{pmatrix} x + \mathrm{i}y & 0 \\ 0 & x - \mathrm{i}y \end{pmatrix} = Q^{-1} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} Q, \qquad \text{with} \qquad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\mathrm{i}\sqrt{2}} & \frac{-1}{\mathrm{i}\sqrt{2}} \end{pmatrix}.$$

Then, remark 2.4 and the previous step ensures that (12) still holds when L is real, semi simple and under real Jordan normal forms. $\square$

The non semi-simple case. For non semi simple operators L the direct computation of the norm $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ is in general quite intricate. So we use the following lemma which gives this norm in terms of the spectrum of the self adjoint operator $(\mathcal{A}_L|_{\mathcal{H}_k})^*\mathcal{A}_L|_{\mathcal{H}_k} = \mathcal{A}_{L^*}|_{\mathcal{H}_k}\mathcal{A}_L|_{\mathcal{H}_k}$ which happens more easy to handle.

Lemma 2.7 *For every linear operator L in $\mathbb{R}^m$ or $\mathbb{C}^m$ and every $k \geq 1$, let us denote $\Sigma_k(L) \subset \mathbb{R}^+$ the spectrum the positive self adjoint operator $(\mathcal{A}_L|_{\mathcal{H}_k})^*\mathcal{A}_L|_{\mathcal{H}_k} = \mathcal{A}_{L^*}|_{\mathcal{H}_k}\mathcal{A}_L|_{\mathcal{H}_k}$. Then,*

$$a_k := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 = \left(\min_{\sigma\in\Sigma_k(L)\setminus\{0\}} |\sigma|\right)^{-\frac{1}{2}}.$$

Proof. Observe that

$$a_k = \sup_{\Phi\in\mathcal{H}_k\setminus\{0\}} \frac{\left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\Phi\right|_2}{|\Phi|_2} = \sup_{\Psi\in(\ker\mathcal{A}_L|_{\mathcal{H}_k})^\perp} \frac{|\Psi|_2}{\left|\mathcal{A}_L|_{\mathcal{H}_k}\Psi\right|_2} = \left(\inf_{\Psi\in(\ker\mathcal{A}_L|_{\mathcal{H}_k})^\perp} \frac{\left\langle \mathcal{A}_{L^*}|_{\mathcal{H}_k}\mathcal{A}_L|_{\mathcal{H}_k}\Psi|\Psi\right\rangle}{|\Psi|_2^2}\right)^{-\frac{1}{2}}.$$

Then, since $\ker\mathcal{A}_L|_{\mathcal{H}_k} = \ker\mathcal{A}_{L^*}|_{\mathcal{H}_k}\mathcal{A}_L|_{\mathcal{H}_k}$ and since $\mathcal{A}_{L^*}|_{\mathcal{H}_k}\mathcal{A}_L|_{\mathcal{H}_k}$ is a positive self adjoint operator, we get $a_k := \left(\min_{\sigma\in\Sigma_k(L)\setminus\{0\}} |\sigma|\right)^{-\frac{1}{2}}$.

□

This lemma enables us to compute the norm of $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for various examples of non semi simple linear operator L. Coupled with Proposition 2.2, these computations gives the proof of Theorem 1.4-(b).

Lemma 2.8

(a) *For $L = \mathbf{0}^2$, $L = \mathbf{0}^3$, $L = \mathbf{0}^2.\mathbf{i}\boldsymbol{\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ and $L = (\mathbf{i}\boldsymbol{\omega})^2|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$, the norm a_k of $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ satisfies*

$$a_k \leq a, \qquad \textit{for every } k \geq 1,$$

respectively with $a_k = 1$, $a_k = 1$, $a = \max(1, \omega^{-1})$ and $a = \max(1, \omega^{-1})$.

(b) *For $L = \mathbf{0}^2.\mathbf{i}\boldsymbol{\omega}_1.\cdots.\mathbf{i}\boldsymbol{\omega}_q|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ and $L = (\mathbf{i}\boldsymbol{\omega}_1)^2.\mathbf{i}\boldsymbol{\omega}_2.\cdots.\mathbf{i}\boldsymbol{\omega}_q|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$, where $\omega := (\omega_1, \cdots, \omega_q, -\omega_1, \cdots, -\omega_q) \in \mathbb{R}^{2q}$ is γ, τ-homologically diophantine, the norm a_k of $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ satisfies*

$$a_k \leq a k^\tau, \qquad \textit{for every } k \geq 1,$$

with $a = \max(2^{-\tau}, \gamma^{-1})$.

The proof of this lemma is postponed to subsection 2.4.

2.3 Exponentially small estimates of the remainder for polynomially bounded pseudo inverse of the homological operator.

This subsection is devoted to the proof of proposition 2.2. To fix the notations we make he proof vector fields in $\mathbb{R}^m$. The proof is the same for $\mathbb{C}^m$. So, let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ such that $V(0) = 0$, i.e. a vector field satisfying (3) and (4). We assume that the pseudo inverse of the homological operator is polynomially bounded on $\mathcal{H}_k$ for $2 \le k \le K \le +\infty$, i.e we assume that there exists $a > 0$ and $\tau \ge 0$ such that

$$a_k = \left| \widetilde{\mathcal{A}_L} |_{\mathcal{H}_k}^{-1} . \Phi \right|_2 \le a k^\tau \qquad \text{for } 2 \le k \le K.$$

Our aim is to find an exponential upper bound of the remainder $\mathcal{R}_p(Y)$ for Y in a ball of radius δ. Since the remainder $\mathcal{R}_p(Y)$ is given by equation (7), for estimating it, we successively compute upper bounds for $\Phi_n(Y)$, $N_n(Y)$, $\sum\limits_{2\le k\le p} D\Phi_k(Y)$, $\sum\limits_{1\le k\le p} \Phi_k(Y)$ and finally for $\mathcal{R}_p(Y)$. For the polynomials N_n and Φ_n the natural norm to finally compute an upper bound of $\sup\limits_{\|Y\|\le\delta} \|\mathcal{R}_p(Y)\|$ is the "sup-norm" defined for any $\Phi \in \mathcal{H}$ by

$$|\Phi|_{0,n} = \sup_{Y\in\mathbb{C}^m} \frac{\|\Phi(Y)\|}{\|Y\|^n}$$

However, N_n and Φ_n are the solution of the Homological Equation (E_n) given by (9), i.e. defined via the orthogonal projector π_n which has nice properties for the euclidian norm and not for the sup norm. These two norm can be compared has follows :

Lemma 2.9 (Comparison of the euclidian and the sup norm)
For every $\Phi \in \mathcal{H}_k$,

$$|\Phi|_{0,k} \le \frac{1}{\sqrt{k!}} |\Phi|_2 \le \sqrt{C_{k+m-1}^{m-1}}\ |\Phi|_{0,k} \le \sqrt{m}\ k^{\frac{m-1}{2}}\ |\Phi|_{0,k}.$$

where $C_n^r = \dfrac{n!}{r!(n-r)!}$.

The proof of this Lemma is given in Appendix A. Moreover if we normalize the euclidian norm on $\mathcal{H}_n$ by defining

$$|\Phi|_{2,n} := \frac{1}{\sqrt{n!}} |\Phi|_2, \qquad \text{for every } \Phi \in \mathcal{H}_n,$$

then the normalized euclidian norm has very nice properties with respect to multiplication and derivation :

Lemma 2.10 (Multiplicativity of the normalized euclidian norm)

(a) *Let q and $\{p_\ell\}_{1\le\ell\le q}$ be positive integers and let $R_q \in \mathcal{L}_q(\mathbb{R}^m)$ be q-linear. Then for every $\Phi_{p_\ell} \in \mathcal{H}_{p_\ell}$, $1 \le \ell \le q$, the polynomial $R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]$ lies in $\mathcal{H}_n$ with $n = p_1 + \cdots + p_q$ and*

$$\left|R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]\right|_{2,n} \le |||R_q|||_{\mathcal{L}_q(\mathbb{R}^m)} \left|\Phi_{p_1}\right|_{2,p_1} \cdots \left|\Phi_{p_q}\right|_{2,p_q}.$$

(b) *Let $k>0$ and $p\geq 0$ be two integers and let Φ_k, N_p lie respectively in $\mathcal{H}_k$ and $\mathcal{H}_p$. Then $D\Phi_k.N_p$ lies in $\mathcal{H}_n$ with $n=k-1+p$ and*

$$|D\Phi_k.N_p|_{2,n} \leq \sqrt{k^2+(m-1)k}\ \ |\Phi_k|_{2,k}\,|N_p|_{2,p}$$

This Lemma is also proved in Appendix A.

Hence to compute by induction upper bounds of Φ_n, N_n defined via π_n, we use the normalized euclidian norms

$$\begin{aligned}\nu_n &= |N_n|_{2,n}\,, \quad \text{for } n\geq 2,\\ \phi_n &= |\Phi_n|_{2,n}\,, \quad \text{for } n\geq 1,\end{aligned}$$

with the convention $\Phi_1(Y)=Y$ and thus $\phi_1=|Y|_{2,1}=\sqrt{m}$. Lemma 2.9 ensures that the same upper bounds will also hold for the sup norms of N_n, Φ_n. Since π_n is orthogonal, we deduce from (9) that

$$\nu_n = |N_n|_{2,n} = |\pi_n(F_n)|_{2,n} = \tfrac{1}{\sqrt{n!}}\,|\pi_n(F_n)|_2 \leq \tfrac{1}{\sqrt{n!}}\,|F_n|_2 = |F_n|_{2,n}$$

and similarly

$$\phi_n \leq |||\widetilde{\mathcal{A}_L}|^{-1}_{\mathcal{H}_n}|||_2\ \ |F_n|_{2,n} \leq an^{\tau}\,|F_n|_{2,n}\,.$$

Hence using the multipicativity and the derivation properties of the normalized euclidian norms, we get that

$$\nu_n \ \leq \sum_{2\leq k\leq n-1}\Big(k^2+(m-1)k\Big)^{\frac{1}{2}}\phi_k\nu_{n-k+1} + \sum_{2\leq q\leq n}\ \sum_{p_1+\cdots+p_q=n}\frac{c}{\rho^q}\phi_{p_1}\cdots\phi_{p_q}, \tag{13}$$

$$\phi_n \ \leq\ an^{\tau}\sum_{2\leq k\leq n-1}\Big(k^2+(m-1)k\Big)^{\frac{1}{2}}\phi_k\nu_{n-k+1} + an^{\tau}\sum_{2\leq q\leq n}\ \sum_{p_1+\cdots+p_q=n}\frac{c}{\rho^q}\phi_{p_1}\cdots\phi_{p_q} \tag{14}$$

for $2\leq n\leq K$ with the convention $\phi_1=|\Phi_1|_{2,1}=|Y|_{2,1}=\sqrt{m}$. Hence using that $(k^2+(m-1)k)^{\frac{1}{2}}\leq\sqrt{m}k$, we check by induction that

Lemma 2.11 *Let $\{\beta_n\}_{n\geq 1}$ be the sequence defined by induction*

$$\begin{aligned}\beta_n &= m\sum_{2\leq k\leq n-1}k\beta_k\beta_{n-k+1} + \sum_{2\leq q\leq n}\ \sum_{p_1+\cdots+p_q=n}\left(\frac{\rho}{ac}\right)^{q-2}\beta_{p_1}\cdots\beta_{p_q}, \quad n\geq 2, \\ \beta_1 &= 1.\end{aligned} \tag{15}$$

Then we have the estimates

$$\nu_n \ \leq\ \frac{\sqrt{m}}{a}\left(\frac{ac\sqrt{m}}{\rho^2}\right)^{n-1}((n-1)!)^{\tau}\beta_n, \quad \text{for } 2\leq n\leq K, \tag{16}$$

$$\phi_n \ \leq\ \sqrt{m}\ \left(\frac{ac\sqrt{m}}{\rho^2}\right)^{n-1}(n!)^{\tau}\beta_n, \quad \text{for } 1\leq n\leq K. \tag{17}$$

Proof. We proceed by induction. For $n = 1$, the above inequality is true since $\phi_1 = \sqrt{m}$. For $n = 2$, equation (15) ensures that $\beta_2 = 1$ and (13), (14) ensure that $\nu_2 \leq cm\rho^{-2}$ and $\phi_2 \leq acm2^\tau \rho^{-2}$, and thus (17), (16) are true for $n = 2$. Assume now that (16), (17) holds for $k < n$ with $n \geq 3$. Then (13) ensures that

$$\nu_n \leq \frac{\sqrt{m}}{a}\left(\frac{ac\sqrt{m}}{\rho^2}\right)^{n-1}((n-1)!)^\tau \left(m \sum_{2\leq k\leq n-1} k\beta_k\beta_{n-k+1}\,(D'_{n,k})^\tau \right.$$
$$\left. + \sum_{2\leq q\leq n}\sum_{p_1+\cdots+p_q=n}\left(\frac{\rho}{ac}\right)^{q-2}\beta_{p_1}\cdots\beta_{p_q}(D_{n,p_1,\cdots,p_q})^\tau\right)$$

where

$$D'_{n,k} = \frac{k!(n-k)!}{(n-1)!} \qquad \text{and} \qquad D_{n,p_1,\cdots,p_q} = \frac{p_1!\cdots p_q!}{(n-1)!}.$$

It remains to prove that $D'_{n,k} \leq 1$ for $2 \leq k \leq n-1$ and that $D_{n,p_1,\cdots,p_q} \leq 1$ for $2 \leq q \leq n,\ p_1 + \cdots + p_q = n,\ p_j \geq 1$, to ensures that (16) holds for n and similarly that (17) holds also for n.

Denoting $C_n^k = \frac{n!}{k!(n-k)!}$ and observing that $C_n^k \geq n$ for $1 \leq k \leq n-1$, we get

$$D'_{n,k} = \frac{n}{C_n^k} \leq 1.$$

Finally to prove that $D_{n,p_1,\cdots,p_q} \leq 1$ we proceed by induction on q. For $q = 2$, we have

$$D_{n,p_1,p_2} = \frac{p_1!(n-p_1)!}{(n-1)!} = D'_{n,p_1} \leq 1$$

since $1 \leq p_1 \leq n-1$. Assume now that $D_{n,p_1,\cdots,p_q} \leq 1$ for $q \geq 2$ and every $n \geq q$, then

$$\begin{aligned} D_{n,p_1,\cdots,p_{q+1}} &= D_{p_1+\cdots+p_q,p_1,\cdots,p_q}\,(p_1+\cdots+p_q-1)!\,\frac{p_{q+1}!}{(n-1)!}, \\ &= D_{p_1+\cdots+p_q,p_1,\cdots,p_q}\,\frac{1}{C_{n-1}^{p_{q+1}}} \leq 1, \end{aligned}$$

since for every $r \in \mathbb{N}$ and j with $0 \leq j \leq r$, we have $C_r^j \geq 1$. This completes the proof of Lemma 2.11.

$\square$

The study of the sequence $\{\beta_n\}_{n\geq 1}$ enables to obtain Gevrey estimates for ϕ_n, ν_n.

Lemma 2.12 *In choosing* $\alpha_1 = 1$ *and*

$$\alpha_n = \Theta^{n-2}(n-2)!, \qquad \textit{for } n \geq 2,$$

and Θ *large enough such that*

$$ac\Theta > \rho, \tag{18}$$

and

$$\frac{\frac{5}{2}m+2}{\Theta} + \frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}} \leq 1, \tag{19}$$

then β_n in (15) satisfies $\beta_n \le \alpha_n$ for $n \ge 1$ and thus

$$\phi_n \le \frac{acm}{\rho^2}\left(\frac{ac\sqrt{m}\ \Theta}{\rho^2}\right)^{n-2} (n!)^\tau\ (n-2)!, \quad \textit{for } 2 \le n \le K, \textit{ and } \phi_1 = \sqrt{m},$$
$$\nu_n \le \frac{cm}{\rho^2}\left(\frac{ac\sqrt{m}\ \Theta}{\rho^2}\right)^{n-2} \left((n-1)!\right)^\tau\ (n-2)!, \quad \textit{for } 2 \le n \le K.$$

Proof. We proceed by induction. We have $\beta_1 = 1 = \alpha_1 \le \alpha_1$ and $\beta_2 = 1 = \alpha_2 \le \alpha_2$. Assume now that $\beta_k \le \alpha_k$ for $k < n$ and $n \ge 3$.

Step 1. Splitting of the bounds. Then by induction hypothesis,

$$\beta_n \le \Delta_n^1 + \Delta_n^2 \tag{20}$$

with

$$\Delta_n^1 = m \sum_{2\le k\le n-1} k\ \alpha_k\ \alpha_{n-k+1} + \sum_{1\le k\le n-1} \alpha_k\ \alpha_{n-k},$$
$$\Delta_n^2 = \sum_{3\le q\le n}\ \sum_{p_1+\ldots p_q=n} \left(\frac{\rho}{ac}\right)^{q-2} \alpha_{p_1}.....\alpha_{p_q}.$$

Step 2. Two auxiliary sums for $\boldsymbol{\Delta_n^1}$. Let us we define

$$S_n = \sum_{2\le k\le n-1} \frac{k(k-2)!(n-k-1)!}{(n-2)!}.$$

We have the following inequality

$$S_n \le 5/2, \text{ for } n \ge 3. \tag{21}$$

This comes from the identity for $n \ge 5$

$$S_{n+1} - S_n = -\sum_{3\le k\le n-2} \frac{k!}{(n-1)...(n-k)} + \frac{n-4}{(n-1)(n-2)}$$

and the fact that for $k = n-2$ the corresponding term in the sum cancels the last positive term. Then a direct calculation of the cases $n = 3$ and $n = 4$ shows the result (21).

We now define

$$P_n = \sum_{2\le k\le n-2} \frac{(k-2)!(n-k-2)!}{(n-2)!}$$

and we check that

$$P_n \le 1/2, \quad \text{for } n \ge 4 \tag{22}$$

since $P_{n+1} - P_n < 0,$ for $n \ge 4$.

Step 3. Upper bound for $\boldsymbol{\Delta_n^1}$. It results from (21) and (22) that for $\Theta \ge 1$,

$$\Delta_n^1 \le \frac{\frac{5}{2}m + 2}{\Theta}\alpha_n, \quad n \ge 3, \tag{23}$$

where the proof of this inequality is direct for $n = 3$.

Step 4. Auxiliary sums for Δ_n^2. Now, we define for $n \geq q \geq 2$:

$$\Pi_{q,n} = \sum_{p_1+\ldots p_q=n} \alpha_{p_1}.....\alpha_{p_q},$$

then we already have

$$\begin{array}{rcl} \Pi_{n,n} & = & 1 \leq \dfrac{1}{\Theta^{n-2}}\alpha_n, \quad n \geq 3 \\ \Pi_{2,2} & = & 1, \\ \Pi_{2,n} & \leq & \dfrac{2}{\Theta}\alpha_n, \quad n \geq 3, \end{array}$$

where the last inequality comes easily from the inequality for P_n. For estimating $\Pi_{q,n}$ with $n \geq q+1$, we proceed as follows

$$\Pi_{q,n} = \sum_{1\leq k\leq n-q+1} \alpha_k \Pi_{q-1,n-k} = \Pi_{q-1,n-1} + \alpha_{n-q+1} + \sum_{2\leq k\leq n-q} \alpha_k \Pi_{q-1,n-k}$$

and prove by recurrence that

$$\Pi_{q,n} \leq \frac{2}{\Theta^{q-1}}\alpha_n, \quad n \geq q+1 \geq 3.$$

Finally, gathering all our results, we get

$$\Pi_{q,n} \leq \frac{2}{\Theta^{q-2}}\alpha_n, \quad n \geq q \geq 3, \tag{24}$$

Step 5. Upper bound for Δ_n^2. We deduce from (24) that

$$\Delta_2^n = \sum_{3\leq q\leq n} \left(\frac{\rho}{ac}\right)^{q-2} \Pi_{q,n} \leq \sum_{3\leq q\leq n} 2\left(\frac{\rho}{ac\Theta}\right)^{q-2} \alpha_n \leq \alpha_n \left\{\frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}}\right\}, \tag{25}$$

provided that $\frac{\rho}{ac\Theta} < 1$.

Step 6. Upper bound for β_n. Hence, (23) and (25) ensure that

$$\beta_n \leq \left\{\frac{\frac{5}{2}m+2}{\Theta} + \frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}}\right\}\alpha_n \leq \alpha_n$$

provided that $\dfrac{\rho}{ac\Theta} < 1$ and $(\frac{5}{2}m+2)\ \dfrac{1}{\Theta} + \dfrac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}} \leq 1.$

□

In all what follows we choose

$$\Theta = \frac{5}{2}m + 2 + \frac{3\rho}{ac} \tag{26}$$

which ensures that (18) and (19) are simultaneously satisfied since with this choice

$$\frac{\rho}{ac\Theta} < \frac{1}{3}, \quad \text{and} \quad (\frac{5}{2}m+2)\ \frac{1}{\Theta} + \frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}} \leq \frac{(\frac{5}{2}m+2)\ \frac{1}{\Theta}}{1-\frac{\rho}{ac\Theta}} + \frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}} = 1.$$

We can now compute an upper bound for the change of coordinates and for its differential.

Lemma 2.13 *For every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying*

$$\delta p^{1+\tau} \leq \frac{\rho^2}{2ac\sqrt{m}\ \Theta}. \tag{27}$$

we have

$$\|\sum_{1\leq k\leq p} \Phi_k(Y)\| \quad \leq \quad \frac{10}{9}\sqrt{m}\ \delta, \tag{28}$$

$$|||\sum_{2\leq k\leq p} D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)} \quad \leq \quad 2/5. \tag{29}$$

for every $Y \in \mathbb{R}^m$ with $||Y|| \leq \delta$.

Remark 2.14 Observe that the size δ of the ball where Y lies and the degree p of the normal form, i.e. the degree of the polynomial change of variable are now mutually constrained by (27).

Proof. We proceed in three steps.

Step 1. Upper bound for $\|\sum_{1\leq k\leq p} \Phi_k(Y)\|$. Lemmas 2.9, 2.12 ensure that

$$\begin{aligned}
\|\sum_{1\leq k\leq p} \Phi_k(Y)\| &\leq \sum_{1\leq k\leq p} |\Phi_k|_{0,k} \|Y\|^k, \\
&\leq \sum_{1\leq k\leq p} |\Phi_k|_{2,k} \|Y\|^k, \\
&\leq \sum_{1\leq k\leq p} \phi_k \delta^k, \\
&\leq \delta\sqrt{m} + \sum_{2\leq k\leq p} \frac{acm\ \delta^2}{\rho^2}\left(\frac{ac\sqrt{m}\Theta\ \delta}{\rho^2}\right)^{k-2} (k!)^\tau\ (k-2)! \\
&\leq \delta\sqrt{m}\left\{1 + \frac{1}{\Theta}\sum_{2\leq k\leq p}\left(\frac{1}{2p^{1+\tau}}\right)^{k-1} (k!)^\tau\ (k-2)!\right\}, \\
&\leq \delta\sqrt{m}\left\{1 + \frac{1}{\Theta\ p}\sum_{2\leq k\leq p}\left(\frac{1}{2}\right)^{k-1}\right\},
\end{aligned}$$

since for $2 \leq k \leq p$,

$$\frac{(k-2)!}{p^{k-1}} \leq \frac{1}{p}, \quad \text{and} \quad \frac{k!}{p^{k-1}} = \frac{2}{p}.\cdots.\frac{k}{p} \leq 1. \tag{30}$$

Hence,

$$\|\sum_{1\leq k\leq p} \Phi_k(Y)\| \leq \delta\sqrt{m}\left\{1 + \frac{1}{p\Theta}\right\} \leq \frac{10}{9}\sqrt{m}\ \delta,$$

since $\Theta \geq \frac{5}{2}m + 2 \geq \frac{9}{2}$ and $p \geq 2$.

Step 2. Upper bound for $|||D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)}$. For $Y, Z \in \mathbb{R}^m$ seeing Z as an homogeneous polynomial of degree 0, Lemmas 2.9, 2.10 ensure that

$$\begin{aligned}
\frac{\|D\Phi_k(Y).Z\|}{\|Y\|^{k-1}} &\leq \left|D\Phi_k(Y).Z\right|_{0,k}, \\
&\leq \left|D\Phi_k(Y).Z\right|_{2,k}, \\
&\leq \sqrt{k^2+(m-1)k}\ \left|\Phi_k\right|_{2,k} \left|Z\right|_{2,0}, \\
&= \sqrt{k^2+(m-1)k}\ \phi_k\ \|Z\|.
\end{aligned}$$

Hence using that $\sqrt{k^2+(m-1)k} \leq \sqrt{m}k$ we obtain

$$|||D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)} \leq \sqrt{m}k\phi_k\|Y\|^{k-1}.$$

Step 3. Upper bound for $|||\sum_{2\leq k\leq p} D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)}$. Lemma 2.12, the previous step and estimate (30) ensure that for $\|Y\| \leq \delta$, with δ, p satisfying (27) we have

$$\begin{aligned}
|||\sum_{2\leq k\leq p} D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)} &\leq \frac{m}{\Theta}\sum_{2\leq k\leq p}\left(\frac{ac\sqrt{m}\Theta\delta}{\rho^2}\right)^{k-1} k\ (k!)^\tau\ (k-2)!, \\
&\leq \frac{m}{\Theta}\sum_{2\leq k\leq p}\left(\frac{1}{2p^{1+\tau}}\right)^{k-1} k\ (k!)^\tau\ (k-2)!, \\
&\leq \frac{m}{\Theta}\sum_{2\leq k\leq p}\left(\frac{1}{2}\right)^{k-1}, \\
&\leq \frac{m}{\Theta}, \\
&\leq \frac{2}{5},
\end{aligned}$$

since $\Theta \geq \frac{5}{2}m$. $\square$

We have now enough material to compute an upper bound of the remainder.

Lemma 2.15 *For every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying*

$$\delta p^{1+\tau} \leq \frac{\rho^2}{e^{1+\tau}ac\sqrt{m}\ \Theta}. \tag{31}$$

we have

$$\|\mathcal{R}_p(Y)\| \leq \frac{10c}{9}\left((C\delta)^{p+1}(p!)^{1+\tau} + \frac{1}{p^{2+2\tau}}\ \left(\frac{1}{e^{1+\tau}}\right)^{p+1}\right)$$

for every $Y \in \mathbb{R}^m$ with $||Y|| \leq \delta$ where

$$C = \frac{ac\sqrt{m}\Theta}{\rho^2} = \frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m+2\right)ac+3\rho\right\}.$$

.

Remark 2.16 Observe that the constraint (31) imposed on δ and p is slightly stronger than the one (27) imposed in Lemma 2.13 since $\frac{1}{e^{1+\tau}} \leq \frac{1}{2}$. The constraint (31) has been chosen to get the optimal exponential decay rate for the upper bound of $\mathcal{R}_p$ obtained by an optimal choice of $p_{\text{opt}} = \left[\frac{1}{\mathrm{e}(C\delta)^{\frac{1}{1+\tau}}}\right]$, i.e $\delta(p_{\text{opt}})^{1+\tau} \approx \frac{1}{\mathrm{e}^{1+\tau}C}$ (for details see below lemmas 2.17 and 2.19).

Proof. The remainder $\mathcal{R}_p(Y)$ is given by equation (7) where it gathers all the terms of order larger than p. To bound it, we proceed in several steps.

Step 1. Splitting of the upper bound. From the explicit expression of $\mathcal{R}_p$ given by (7) and using lemmas 2.9, 2.10, 2.13, we get that for every $\delta > 0$, every p, $2 \leq p \leq K$ satisfying (27) and for every $Y \in \mathbb{R}^m$ with $\|Y\| \leq \delta$,

$$\frac{3}{5}\|\mathcal{R}_p(Y)\| \leq \Delta_p^1 + \Delta_p^2 + \Delta_p^3 \tag{32}$$

where

$$\begin{aligned}
\Delta_p^1 &= \sum_{\substack{2\leq k\leq p \\ p+1\leq n\leq p+k-1}} \sqrt{m}\ k\phi_k\nu_{n-k+1}\delta^n \\
\Delta_p^2 &= \sum_{\substack{2\leq q\leq p \\ p+1\leq n=p_1+\cdots+p_q, \\ 1\leq p_j\leq p}} \frac{c\delta^n}{\rho^q}\phi_{p_1}\cdots\phi_{p_q} \\
\Delta_p^3 &= \sum_{p+1\leq q} \frac{c}{\rho^q}\left(\frac{10}{9}\sqrt{m}\ \delta\right)^q.
\end{aligned}$$

The sums Δ_p^1 and Δ_p^3 can be optimally bounded with constraint (27) whereas for Δ_p^2 we use the stronger constraint (31).

Step 2. Upper bound for $\boldsymbol{\Delta_p^1}$. Defining $C = \frac{ac\sqrt{m}\ \Theta}{\rho^2}$ and using lemma 2.12 we get

$$\begin{aligned}
\Delta_p^1 &\leq m^{\frac{5}{2}}\frac{ac^2}{\rho^4}\sum_{\substack{2\leq k\leq p \\ p+1\leq n\leq p+k-1}}\left(\frac{ac\sqrt{m}\ \Theta}{\rho^2}\right)^{n-3}\delta^n\ k(k!)^\tau(k-2)!\ \left((n-k)!\right)^\tau\ (n-k-1)!, \\
&\leq \frac{m\rho^2}{a^2c\ \Theta^3}\sum_{2\leq k\leq p} k\ (k!)^\tau\ (k-2)!\ \ (C\delta)^{p+1}\sum_{p+1\leq n\leq p+k-1}(C\delta)^{n-p-1}(n-k-1)!\ \left((n-k)!\right)^\tau, \\
&\leq \frac{m\rho^2}{a^2c\ \Theta^3}\sum_{2\leq k\leq p} k(k!)^\tau(k-2)!\ (C\delta)^{p+1}\sum_{p+1\leq n\leq p+k-1}\left(\tfrac{1}{2}\right)^{n-p-1}\frac{(n-k-1)!}{p^{n-p-1}}\left(\frac{(n-k)!}{p^{n-p-1}}\right)^\tau,
\end{aligned}$$

since $C\delta \leq \frac{1}{(ep)^{1+\tau}} \leq \frac{1}{2\ p^{1+\tau}}$ (here we do not need the strongest constraint). Then, observe that for $p+1 \leq n \leq p+k-1$,

$$\frac{(n-k-1)!}{(p-2)^{n-p-1}} \leq (p-k)! \qquad \text{and} \qquad \frac{(n-k)!}{p^{n-p-1}} \leq (p-k+1)!.$$

Thus, we obtain

$$\begin{aligned}
\Delta_p^1 &\le \frac{m\rho^2}{a^2c\ \Theta^3} \sum_{2\le k\le p} k(k!)^\tau (k-2)!\ \ (C\delta)^{p+1}\ 2(p-k)!\big((p-k+1)!\big)^\tau, \\
&\le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \sum_{2\le k\le p} \frac{1}{C_p^k(k-1)} \left(\frac{p+1}{C_{p+1}^k}\right)^\tau, \\
&\le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \sum_{2\le k\le p} \frac{1}{C_p^k(k-1)}, \\
&\le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \sum_{2\le k\le p} \frac{1}{p-1}.
\end{aligned}$$

Hence, for every $\delta > 0$ and every p, $2 \le p \le K$ satisfying (27),

$$\Delta_p^1 \le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \tag{33}$$

Step 3. Upper bound for $\boldsymbol{\Delta_p^2}$. Observing that $\alpha_n \le (n-2)!\ \Theta^{n-1}$ for any $n \ge 1$ where $(-1)! = 0! = 1$ and using Lemma 2.12 we get

$$\begin{aligned}
\Delta_p^2 &= \sum_{2\le q\le p}\ \sum_{n\ge p+1}\ \sum_{\substack{p_1+\cdots+p_q=n\\ 1\le p_j\le p}} \frac{c(\sqrt{m})^q}{\rho^q}\left(\frac{ac\sqrt{m}}{\rho^2}\right)^{n-q} \delta^n\ \big((p_1!)^\tau\alpha_{p_1}\big)\cdots\big((p_q!)^\tau\alpha_{p_q}\big) \\
&\le \sum_{2\le q\le p}\ \sum_{n\ge p+1}\ \sum_{\substack{p_1+\cdots+p_q=n\\ 1\le p_j\le p}} \frac{c(\sqrt{m})^q}{\rho^q}\left(\frac{ac\sqrt{m}\Theta}{\rho^2}\right)^{n-q} \delta^n\ (p_1!)^\tau(p_1-2)!\cdots(p_q!)^\tau(p_q-2)! \\
&\le \sum_{2\le q\le p}\ \sum_{n\ge p+1}\ \sum_{\substack{p_1+\cdots+p_q=n\\ 1\le p_j\le p}} c\left(\frac{ac\sqrt{m}\delta\Theta}{\rho^2}\right)^{n}\left(\frac{\rho}{ac\Theta}\right)^q (p_1!)^\tau(p_1-2)!\cdots(p_q!)^\tau(p_q-2)!, \\
&\le c\sum_{2\le q\le p} r^q \sum_{n\ge p+1}\ \sum_{\substack{p_1+\cdots+p_q=n\\ 1\le p_j\le p}} (C\delta)^n\ (p_1!)^\tau(p_1-2)!\cdots(p_q!)^\tau(p_q-2)!,
\end{aligned}$$

since $C = \frac{ac\sqrt{m}\ \Theta}{\rho^2}$ and where $r := \frac{\rho}{ac\Theta} \le \frac{1}{3}$ with our choice of Θ given by (26). Moreover, for $\delta > 0$ and $p \ge 2$ satisfying (31) (here we use the stronger constraint), i.e. for $C\delta \le \frac{1}{(ep)^{1+\tau}}$, we obtain

$$
\begin{aligned}
\Delta_p^2 &\leq c \sum_{2\leq q\leq p} r^q \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} \left(\frac{1}{ep}\right)^{n(1+\tau)} (p_1!)^\tau (p_1-2)! \cdots (p_q!)^\tau (p_q-2)!,\\
&\leq c \left(\frac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} r^q \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} \left(\frac{1}{p}\right)^{n(1+\tau)} (p_1!)^\tau (p_1-2)! \cdots (p_q!)^\tau (p_q-2)!,\\
&\leq c \left(\frac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} r^q \left(\sum_{j=1}^{p} \left(\frac{1}{p^{1+\tau}}\right)^j (j!)^\tau (j-2)!\right)^q,\\
&\leq c \left(\frac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} r^q \left(\frac{1}{p^{1+\tau}} + \frac{p-1}{p^{\tau+2}}\right)^q,\\
&\leq c \left(\frac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} \left(\frac{2r}{p^{1+\tau}}\right)^q,\\
&\leq c \left(\frac{1}{e^{1+\tau}}\right)^{p+1} \frac{4r^2}{p^{2+2\tau}} \frac{1}{1-\frac{2r}{p^{1+\tau}}},
\end{aligned}
$$

since $\frac{2}{p^{1+\tau}} \leq 1$. *Hence, for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying* (31),

$$
\Delta_p^2 \leq 4c \left(\frac{\rho}{ac\Theta}\right)^2 \frac{1}{1-\frac{\rho}{ac\Theta}} \frac{1}{p^{2+2\tau}} \left(\frac{1}{e^{1+\tau}}\right)^{p+1}. \tag{34}
$$

Step 4. Upper bound for $\boldsymbol{\Delta_p^3}$. Observing that with our choice of Θ given by (26), for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying (27), we obtain

$$
\frac{\sqrt{m}\ \delta}{\rho} \leq \frac{\rho}{2ac\Theta p^{1+\tau}} \leq \frac{1}{12}
$$

and thus,

$$
\Delta_p^3 = c \sum_{p+1\leq q} \left(\frac{10}{9}\frac{\sqrt{m}\ \delta}{\rho}\right)^q \leq c \left(\frac{10}{9}\frac{\sqrt{m}\ \delta}{\rho}\right)^{p+1} \sum_{q\geq 3} \left(\frac{5}{54}\right)^q.
$$

Hence, for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying (27),

$$
\Delta_p^3 \leq \frac{54}{49} \left(\frac{5}{54}\right)^3 c \left(\frac{10}{9}\frac{\sqrt{m}\ \delta}{\rho}\right)^{p+1}. \tag{35}
$$

Step 5. Upper bound for $\|\boldsymbol{\mathcal{R}_p(Y)}\|$. Gathering the upper bounds for Δ_p^1, Δ_p^2, Δ_p^3 given by (33), (34), (35), that with our choice of Θ given by (26),

$$
\frac{\rho}{ac\Theta} \leq \frac{1}{3}, \qquad \frac{m}{\Theta} \leq \frac{2}{5}
$$

we obtain that for every $\delta > 0$ and every p, $2 \le p \le K$ satisfying (31)

$$\begin{aligned}
\|\mathcal{R}_p(Y)\| &\le \frac{5}{3}\left(\Delta_p^1 + \Delta_p^2 + \Delta_p^3\right) \\
&\le \frac{4c}{27}\,(C\delta)^{p+1}(p!)^{1+\tau} + \frac{10c}{9}\frac{1}{p^{2+2\tau}}\,\left(\frac{1}{e^{1+\tau}}\right)^{p+1} + \frac{90c}{49}\left(\frac{5}{54}\right)^3\left(\frac{10}{9}\frac{\sqrt{m}\,\delta}{\rho}\right)^{p+1} \\
&\le \left(\frac{4}{27} + \frac{90}{49}\left(\frac{5}{54}\right)^3\right)\,c\,(C\delta)^{p+1}(p!)^{1+\tau} + \frac{10c}{9}\frac{1}{p^{2+2\tau}}\,\left(\frac{1}{e^{1+\tau}}\right)^{p+1}
\end{aligned}$$

since with our choice of Θ given by (26),

$$\frac{10\sqrt{m}}{9\rho} = \frac{10}{9}\frac{\rho}{ac\Theta}C \le \frac{10}{27}C \le C.$$

Hence, since $\left(\frac{4}{27} + \frac{90}{49}\left(\frac{5}{54}\right)^3\right) \le \frac{10}{9}$, for every $\delta > 0$ and every p, $2 \le p \le K$ satisfying (31) we have

$$\|\mathcal{R}_p(Y)\| \le \frac{10}{9}c\left((C\delta)^{p+1}(p!)^{1+\tau} + \frac{1}{p^{2+2\tau}}\,\left(\frac{1}{e^{1+\tau}}\right)^{p+1}\right)$$

for every $Y \in \mathbb{R}^m$ with $||Y|| \le \delta$. $\square$

The upper bound of $\|\mathcal{R}_p(Y)\|$ contains two terms. The second one, $\frac{1}{p^{2+2\tau}}\,\left(\frac{1}{e^{1+\tau}}\right)^{p+1}$ tends to 0 as p tends to infinity whereas the first one $(C\delta)^{p+1}(p!)^{1+\tau}$ tends to infinity. The key idea is to choose an optimal p for which $(C\delta)^{p+1}(p!)^{1+\tau} = \left((C\delta)^{\frac{p+1}{1+\tau}}p!\right)^{1+\tau}$ is minimal and prove that this minimal value is exponentially small with respect to δ. This results from the following lemma :

Lemma 2.17 *Choose $\varepsilon > 0$ and let us define $f_\varepsilon(p) := \varepsilon^{p+1}p!$ for $p \in \mathbb{N}$. Moreover, for $x \in \mathbb{R}$, denote by $[x]$ its entire part.*
Then, for $p_{\rm opt} := \left[\frac{1}{\varepsilon e}\right]$, $f_\varepsilon(p_{\rm opt})$ is exponentially small with respect to ε. Indeed,

$$f\left(\left[\frac{1}{\varepsilon \mathrm{e}}\right]\right) \le \mathfrak{m}\sqrt{\frac{\varepsilon}{\mathrm{e}}}\ \mathrm{e}^{-\frac{2}{\varepsilon\mathrm{e}}}$$

where $\mathfrak{m} = \sup\limits_{p\in\mathbb{N}} \dfrac{\mathrm{e}^2\ p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$.

Remark 2.18 Stirling's formula ensure that M is finite.

Proof.

$$\begin{aligned}
f\left(\left[\frac{1}{\varepsilon\mathrm{e}}\right]\right) &\le \frac{\mathfrak{m}\varepsilon}{\mathrm{e}^2}\exp\left\{\left(\left[\frac{1}{\varepsilon\mathrm{e}}\right] + \frac{1}{2}\right)\ln\left[\frac{1}{\varepsilon\mathrm{e}}\right] + \left[\frac{1}{\varepsilon\mathrm{e}}\right]\ln\frac{\varepsilon}{\mathrm{e}}\right\}, \\
&\le \frac{\mathfrak{m}\varepsilon}{\mathrm{e}^2}\exp\left\{\left(\left[\frac{1}{\varepsilon\mathrm{e}}\right] + \frac{1}{2}\right)\ln\frac{1}{\varepsilon\mathrm{e}} + \left[\frac{1}{\varepsilon\mathrm{e}}\right]\ln\frac{\varepsilon}{\mathrm{e}}\right\}, \\
&= \frac{\mathfrak{m}\varepsilon}{\sqrt{\varepsilon\mathrm{e}}}\exp\left\{-2\left(\left[\frac{1}{\varepsilon\mathrm{e}}\right] + 1\right)\right\} \le \mathfrak{m}\sqrt{\frac{\varepsilon}{\mathrm{e}}}\ \mathrm{e}^{-\frac{2}{\varepsilon\mathrm{e}}}.
\end{aligned}$$

$\square$

Using this lemma we finally obtain the desired exponentially small upper bound for $\mathcal{R}_p(Y)$.

Lemma 2.19 *If there exits $K \geq 2$, $a > 0$ and $\tau \geq 0$ such that $a_k := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq a k^\tau$ for every k with $2 \leq k \leq K$, then for every $\delta > 0$ such that $K \geq p_{\text{opt}} \geq 2$, the remainder $\mathcal{R}_p$ given by the Normal Form Theorem 1.1 for $p = p_{\text{opt}}$ satisfies*

$$\sup_{\|Y\| \leq \delta} \|R_{p_{\text{opt}}}(Y)\| \leq M\delta^2 \exp\left(-\tfrac{w}{\delta^b}\right)$$

with

$$b = \frac{1}{1+\tau}, \quad p_{\text{opt}} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right], \quad w = \frac{1}{\mathrm{e}C^b}, \quad M = \frac{10}{3} c C^2 \left\{ \left(\mathfrak{m}\sqrt{\frac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau} \right\}$$

where $C = \frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m + 2\right) ac + 3\rho\right\}$ and $\mathfrak{m} = \sup_{p\in\mathbb{N}} \dfrac{\mathrm{e}^2\, p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$.

Proof. Let $\delta > 0$ be such that $p_{\text{opt}} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right]$ satisfies $K \geq p_{\text{opt}} \geq 2$. Observe that condition (31) reads $\delta^b p \leq \frac{1}{\mathrm{e}C^b}$ and thus that p_{opt} satisfies it. Then since,

$$p_{\text{opt}} + 1 \geq \frac{1}{\mathrm{e}(C\delta)^b} \geq p_{\text{opt}} \geq 2 \qquad \text{and} \qquad \frac{1}{p_{\text{opt}}} \leq 2\ \mathrm{e}(C\delta)^b$$

lemmas 2.15 and 2.17 with $\varepsilon = (C\delta)^b$ ensure that

$$\begin{aligned}
\sup_{\|Y\|\leq\delta} \|R_{p_{\text{opt}}}(Y)\| &\leq \frac{10}{9} c \left\{ \left(\mathfrak{m}\sqrt{\frac{(C\delta)^b}{\mathrm{e}}}\mathrm{e}^{-\frac{2}{\mathrm{e}(C\delta)^b}} \right)^{1+\tau} + \left(2\mathrm{e}(C\delta)^b\right)^{2+2\tau}\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}} \right\}, \\
&\leq \frac{10}{9} c(\mathrm{e}^{1+\tau}C\delta)^2\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}} \left\{ \left(\frac{\mathfrak{m}}{\mathrm{e}}(\mathrm{e}(C\delta)^b)^{-\frac{3}{2}}\mathrm{e}^{-\frac{1}{\mathrm{e}(C\delta)^b}} \right)^{1+\tau} + 4^{1+\tau} \right\}, \\
&\leq \frac{10}{9} c(\mathrm{e}^{1+\tau}C\delta)^2\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}} \left\{ \left(\frac{\mathfrak{m}}{\mathrm{e}}\sqrt{\frac{27}{8}}\mathrm{e}^{-\frac{3}{2}} \right)^{1+\tau} + 4^{1+\tau} \right\}, \\
&= \frac{10}{9} c C^2 \left\{ \left(\mathfrak{m}\sqrt{\frac{27}{8\mathrm{e}}} \right)^{1+\tau} + (2\mathrm{e})^{2+2\tau} \right\}\ \delta^2\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}},
\end{aligned}$$

since $x^{\frac{3}{2}}\mathrm{e}^{-x} \leq \sqrt{\frac{27}{8}}\mathrm{e}^{-\frac{3}{2}}$ for any $x \geq 0$. $\square$

2.4 Computations of the norm of the pseudo inverse of the homological operator for non semi simple-matrices.

This subsection is devoted to the computation of the norm of $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for various examples of non semi simple operator L. We begin with the $\mathbf{0^2}$ singularity. All these computations of the norm of the pseudo inverse of the homological operator are performed via lemma 2.7. Hence, in all this subsection we denote by $\Sigma_k(L) \subset \mathbb{R}^+$ the spectrum of the positive self adjoint operator $(\mathcal{A}_L|_{\mathcal{H}_k})^* \mathcal{A}_L|_{\mathcal{H}_k} = \mathcal{A}_{L^*}|_{\mathcal{H}_k} \mathcal{A}_L|_{\mathcal{H}_k}$.

Lemma 2.20 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for $L = \mathbf{0^2}$)
For $L = \mathbf{0^2}$ and for every $k \geq 2$, we have $\min_{\lambda \in \Sigma_k(L) \setminus \{0\}} \{\lambda\} \geq 1$ *and thus*

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

Proof. We are in dimension 2, with $Y = (x, y)$ and $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in the subspace $\mathcal{H}_k$ of homogeneous polynomials of degree k. We recall that

$$\mathcal{A}_L \Phi(Y) = D\Phi(Y)LY - L\Phi(Y).$$

Thus, denoting $\Phi = (\phi_1, \phi_2)$ in $\mathcal{H}_k$, we have

$$\mathcal{A}_L \Phi = (y\frac{\partial \phi_1}{\partial x} - \phi_2, y\frac{\partial \phi_2}{\partial x}) \qquad \text{and} \qquad \ker \mathcal{A}_L = \text{span}\{(y^k, 0), (xy^{k-1}, y^k)\}.$$

Now we look for the eigenvalues λ ($\lambda \geq 0$) of $\mathcal{A}_{L^*}\mathcal{A}_L$ in the subspace $\mathcal{H}_k$. They are given by

$$\begin{array}{rcl} xy\dfrac{\partial^2 \phi_1}{\partial x \partial y} + x\dfrac{\partial \phi_1}{\partial x} - x\dfrac{\partial \phi_2}{\partial y} & = & \lambda \phi_1 \\ xy\dfrac{\partial^2 \phi_2}{\partial x \partial y} + x\dfrac{\partial \phi_2}{\partial x} - y\dfrac{\partial \phi_1}{\partial x} + \phi_2 & = & \lambda \phi_2. \end{array} \tag{36}$$

We check that
i) $\Phi = (0, x^k)$ gives $\lambda = k + 1$
ii) $\Phi = (y^k, 0)$ gives $\lambda = 0$
iii) $\Phi = (x^\alpha y^\beta, x^{\alpha-1}y^{\beta+1})$ gives $\lambda = (\alpha - 1)(\beta + 1)$ with $\alpha + \beta = k$, $\alpha = 1, ...k$
iv) $\Phi = ((\beta + 1)x^\alpha y^\beta, -\alpha x^{\alpha-1}y^{\beta+1})$ gives $\lambda = \alpha(\beta + 2)$ with $\alpha + \beta = k$, $\alpha = 1, ...k$.

These are the $2(k + 1)$ eigenvalues of the operator $\mathcal{A}^*\mathcal{A}$ in the subspace $\mathcal{H}_k$, corresponding to a family of orthogonal eigenvectors. It is clear that $\min_{\lambda \in \Sigma_k(L) \setminus \{0\}} \{\lambda\} \geq 1$ and thus,

$$a_k := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

□

Lemma 2.21 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for $L = \mathbf{0^3}$)
For $L = \mathbf{0^3}$ and for every $k \geq 2$, we have $\min_{\lambda \in \Sigma_k(L) \setminus \{0\}} \{\lambda\} \geq 1$ *and thus*

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

Proof. We are in dimension 3, with $Y = (x, y, z)$, $\Phi = (\phi_1, \phi_2, \phi_3)$ and

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here again, we intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$. This is performed in several steps.

Step 1. Splitting of the operators. We define differential operators $\mathcal{D}$ and $\mathcal{D}^*$ by

$$\begin{aligned} \mathcal{D} &= y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \\ \mathcal{D}^* &= x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \end{aligned}$$

then

$$\begin{aligned} \mathcal{A}_L\Phi &= \mathcal{D}\Phi - L\Phi \\ \mathcal{A}_{L^*}\Psi &= \mathcal{D}^*\Psi - L^*\Psi, \end{aligned}$$

and

$$\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \begin{pmatrix} \mathcal{D}^*(\mathcal{D}\phi_1 - \phi_2) \\ \mathcal{D}^*(\mathcal{D}\phi_2 - \phi_3) - \mathcal{D}\phi_1 + \phi_2 \\ \mathcal{D}^*\mathcal{D}\phi_3 - \mathcal{D}\phi_2 + \phi_3 \end{pmatrix}.$$

Moreover, we check that $\ker \mathcal{A}_L$ is spanned by

$$\begin{pmatrix} z^\alpha(xz - \frac{y^2}{2})^\beta \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} yz^\alpha(xz - \frac{y^2}{2})^\beta \\ z^{\alpha+1}(xz - \frac{y^2}{2})^\beta \\ 0 \end{pmatrix}, \begin{pmatrix} xz^\alpha(xz - \frac{y^2}{2})^\beta \\ yz^\alpha(xz - \frac{y^2}{2})^\beta \\ z^{\alpha+1}(xz - \frac{y^2}{2})^\beta \end{pmatrix}.$$

In what follows we use the properties

$$\begin{aligned} \mathcal{D}x &= y, \;\; \mathcal{D}y = z, \;\; \mathcal{D}z = 0, \; \mathcal{D}(xz - \frac{y^2}{2}) = 0, \\ \mathcal{D}^*x &= 0, \;\; \mathcal{D}^*y = x, \;\; \mathcal{D}^*z = y, \; \mathcal{D}^*(xz - \frac{y^2}{2}) = 0. \end{aligned}$$

Step 2. Splitting of $\mathcal{H}_k$. Using the basis of monomials, for α, β, γ integers ≥ 0

$$\phi_{\alpha,\beta,\gamma} = x^\alpha z^\beta (xz - \frac{y^2}{2})^\gamma, \qquad \text{and} \qquad \psi_{\alpha,\beta,\gamma} = x^\alpha y z^\beta (xz - \frac{y^2}{2})^\gamma.$$

we split $\mathcal{H}_k$ into the direct sum

$$\mathcal{H}_k = \mathcal{H}'_k \oplus \mathcal{H}''_k$$

where

$$\begin{aligned} \mathcal{H}'_k &= \left\{ \Phi = (\phi_1, \phi_2, \phi_3)/\phi_1, \phi_3 \in \operatorname*{span}_{\alpha+\beta+2\gamma=k} \{\phi_{\alpha,\beta,\gamma}\}, \; \phi_2 \in \operatorname*{span}_{\alpha+\beta+2\gamma+1=k} \{\psi_{\alpha,\beta,\gamma}\} \right\}, \\ \mathcal{H}''_k &= \left\{ \Phi = (\phi_1, \phi_2, \phi_3)/\phi_1, \phi_3 \in \operatorname*{span}_{\alpha+\beta+2\gamma+1=k} \{\psi_{\alpha,\beta,\gamma}\}, \; \phi_2 \in \operatorname*{span}_{\alpha+\beta+2\gamma=k} \{\phi_{\alpha,\beta,\gamma}\} \right\}. \end{aligned}$$

Then, using the identities

$$
\begin{aligned}
\mathcal{D}\phi_{\alpha,\beta,\gamma} &= \alpha\psi_{\alpha-1,\beta,\gamma}, \\
\mathcal{D}\psi_{\alpha,\beta,\gamma} &= (1+2\alpha)\phi_{\alpha,\beta+1,\gamma} - 2\alpha\phi_{\alpha-1,\beta,\gamma+1} \\
\mathcal{D}^*\phi_{\alpha,\beta,\gamma} &= \beta\psi_{\alpha,\beta-1,\gamma}, \\
\mathcal{D}^*\psi_{\alpha,\beta,\gamma} &= (1+2\beta)\phi_{\alpha+1,\beta,\gamma} - 2\beta\phi_{\alpha,\beta-1,\gamma+1}, \\
\mathcal{D}^*\mathcal{D}\phi_{\alpha,\beta,\gamma} &= \alpha(1+2\beta)\phi_{\alpha,\beta,\gamma} - 2\alpha\beta\phi_{\alpha-1,\beta-1,\gamma+1}, \\
\mathcal{D}^*\mathcal{D}\psi_{\alpha,\beta,\gamma} &= (2\alpha+1)(\beta+1)\psi_{\alpha,\beta,\gamma} - 2\alpha\beta\psi_{\alpha-1,\beta-1,\gamma+1},
\end{aligned}
$$

we observe that $\mathcal{H}'_k$ and $\mathcal{H}''_k$ are both invariant under $\mathcal{A}_{L^*}\mathcal{A}_L$. *Hence, the spectrum of the operator $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$ is the union of its spectrum when restricted to $\mathcal{H}'_k$ and to $\mathcal{H}''_k$.*

Step 3. Spectrum of $\boldsymbol{\mathcal{A}_{L^*}\mathcal{A}_L}$ in $\boldsymbol{\mathcal{H}'_k}$. We also split $\mathcal{H}'_k$ into subspaces invariant by $\mathcal{A}_{L^*}\mathcal{A}_L$.

Step 3.1. Splitting of $\boldsymbol{\mathcal{H}'_k}$. First observe that for $\alpha+\beta+2\gamma=k$, the subspace $\mathcal{E}'_{\alpha,\beta,\gamma}$ of $\mathcal{H}'_k$ gathering the polynomials Φ of the form

$$
\begin{aligned}
\phi_1 &= \sum_p a_p\phi_{\alpha-p,\beta-p,\gamma+p}, \\
\phi_2 &= \sum_p b_p\psi_{\alpha-p-1,\beta-p,\gamma+p}, \\
\phi_3 &= \sum_p c_p\phi_{\alpha-p-1,\beta-p+1,\gamma+p},
\end{aligned}
$$

where

$$
\begin{array}{lll}
\text{for } \alpha\le\beta, & 0\le p\le\alpha, & b_\alpha=c_\alpha=0, \\
\text{for } \beta+1\le\alpha, & 0\le p\le\beta+1, & a_{\beta+1}=b_{\beta+1}=0 \quad \text{and } c_{\beta+1}=0 \text{ if } \alpha=\beta+1.
\end{array}
$$

is invariant under the operator $\mathcal{A}_{L^*}\mathcal{A}_L$. Indeed, we have

$$
\begin{aligned}
\mathcal{D}\phi_1-\phi_2 &= \sum\{(\alpha-p)a_p-b_p\}\psi_{\alpha-p-1,\beta-p,\gamma+p}, \\
\mathcal{D}\phi_2-\phi_3 &= \sum\{(2\alpha-2p-1)b_p-c_p\}\phi_{\alpha-p-1,\beta-p+1,\gamma+p} \\
&\qquad -2(\alpha-p-1)b_p\phi_{\alpha-p-2,\beta-p,\gamma+p+1}, \\
\mathcal{D}^*(\mathcal{D}\phi_1-\phi_2) &= \sum(2\beta-2p+1)\{(\alpha-p)a_p-b_p\}\phi_{\alpha-p,\beta-p,\gamma+p} \\
&\qquad -2(\beta-p)\{(\alpha-p)a_p-b_p\}\phi_{\alpha-p-1,\beta-p-1,\gamma+p+1}, \\
\mathcal{D}^*(\mathcal{D}\phi_2-\phi_3) &= \sum(\beta-p+1)\{(2\alpha-2p-1)b_p-c_p\}\psi_{\alpha-p-1,\beta-p,\gamma+p} \\
&\qquad -2(\beta-p)(\alpha-p-1)b_p\psi_{\alpha-p-2,\beta-p-1,\gamma+p+1}, \\
\mathcal{D}^*\mathcal{D}\phi_3 &= \sum(\alpha-p-1)(2\beta-2p+3)c_p\phi_{\alpha-p-1,\beta-p+1,\gamma+p} \\
&\qquad -2(\alpha-p-1)(\beta-p+1)c_p\phi_{\alpha-p-2,\beta-p,\gamma+p+1}.
\end{aligned}
$$

Moreover, $\Phi'_k=(0,0,\phi_{k,0,0})$ is an eigenvector of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$ belonging to the eigenvalue $\lambda=k+1$.

Then, since $\Phi=(\phi_{\alpha,\beta,\gamma},0,0)$, $\Phi=(0,\psi_{\alpha-1,\beta,\gamma},0)$, $\Phi=(0,0,\phi_{\alpha-1,\beta+1,\gamma})$ and $\Phi=(0,0,\phi_{\alpha-2,\beta,\gamma+1})$ belong to $\mathcal{E}'_{\alpha,\beta,\gamma}$ respectively for $\alpha\ge 0$, $\alpha\ge 1$, $\alpha\ge 1$ and $\alpha\ge 2$, we have the splitting of $\mathcal{H}'_k$ into the non direct sum

$$\mathcal{H}'_k=\mathbb{C}\Phi'_k+\sum_{\alpha+\beta+2\gamma=k}\mathcal{E}'_{\alpha,\beta,\gamma}.$$

Hence, the spectrum $\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_{\mathrm{k}}})$ *of the operator* $\mathcal{A}_{L^*}\mathcal{A}_L$ *in* $\mathcal{H}_k$ *is given by the union with possibly many overlaps*

$$\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_{\mathrm{k}}}) = \{\mathrm{k}+1\} \cup \bigcup_{\alpha+\beta+2\gamma=\mathrm{k}} \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{E}'_{\alpha,\beta,\gamma}}).$$

Step 3.2. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}'_{\alpha,\beta,\gamma}$. The spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$, for $\Phi \in \mathcal{E}'_{\alpha,\beta,\gamma}$ can be written as a hierarchy of systems of equation $(37)_p$ where for $p=0$ we have

$$\begin{array}{rcl} (2\beta+1)(\alpha a_0 - b_0) & = & \lambda a_0, \\ (\beta+1)\{(2\alpha-1)b_0 - c_0\} + b_0 - \alpha a_0 & = & \lambda b_0, \\ (\alpha-1)(2\beta+3)c_0 + c_0 - (2\alpha-1)b_0 & = & \lambda c_0, \end{array} \tag{$37)_0$}$$

and for $1 \le p \le \min\{\alpha, \beta+1\}$,

$$\begin{array}{rcl} \lambda a_p & = & (2\beta-2p+1)\{(\alpha-p)a_p - b_p\} \\ & & -2(\beta-p+1)\{(\alpha-p+1)a_{p-1} - b_{p-1}\} \\ \lambda b_p & = & (\beta-p+1)\{(2\alpha-2p-1)b_p - c_p\} - (\alpha-p)a_p + b_p \\ & & -2(\beta-p+1)(\alpha-p)b_{p-1}, \\ \lambda c_p & = & (\alpha-p-1)(2\beta-2p+3)c_p - (2\alpha-2p-1)b_p + c_p \\ & & -2(\alpha-p)(\beta-p+2)c_{p-1} + 2(\alpha-p)b_{p-1}. \end{array} \tag{$37)_p$}$$

In particular, when $\alpha \le \beta$ the last system of the hierarchy is obtained for $p=\alpha$ ($b_\alpha = c_\alpha = 0$) and it reads

$$\begin{array}{rcl} \lambda a_\alpha & = & -2(\beta-\alpha+1)(a_{\alpha-1} - b_{\alpha-1}), \\ 0 & = & 0, \end{array} \tag{$37)_\alpha$}$$

while for $\beta \le \alpha-1$ the last system is obtained for $p=\beta+1$ ($a_{\beta+1} = b_{\beta+1} = 0$, and $c_{\beta+1}=0$ if $\alpha=\beta+1$) and it reads

$$\begin{array}{rcl} \lambda c_{\beta+1} & = & (\alpha-\beta-1)c_{\beta+1} - 2(\alpha-\beta-1)\{c_\beta - b_\beta\}, \\ 0 & = & 0. \end{array} \tag{$37)_{\beta+1}$}$$

The system with $p=0$ gives the eigenvalues:

$$\begin{array}{ll} \lambda_1 = (\alpha-1)(2\beta+1), & a_0 = b_0 = c_0 = 1, \\ \lambda_2 = \alpha(2\beta+3), & a_0 = (\beta+1)(2\beta+1),\ b_0 = -2\alpha(\beta+1),\ c_0 = \alpha(2\alpha-1), \\ \lambda_3 = (2\alpha-1)(\beta+1), & a_0 = -(2\beta+1),\ b_0 = \alpha-\beta-1,\ c_0 = 2\alpha-1. \end{array}$$

We check that for for $\alpha = 0$ or 1, we recover known eigenvectors belonging to the 0 eigenvalue, all other eigenvalues are positive integers.

For proving that they indeed give eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ it is needed to check that for $1 \le p < \min\{\alpha, \beta+1\}$ the determinant Δ_p does not cancel for $\lambda = \lambda_1$ or λ_2 or λ_3 where

$$\Delta_p = \begin{vmatrix} (2\beta'+1)\alpha' - \lambda & -(2\beta'+1) & 0 \\ -\alpha' & (\beta'+1)(2\alpha'-1)+1-\lambda & -(\beta'+1) \\ 0 & -2\alpha'+1 & (\alpha'-1)(2\beta'+3)+1-\lambda \end{vmatrix}$$

with $\alpha' = \alpha - p$, $\beta' = \beta - p$. It results that

$$\Delta_p = (\lambda'_1 - \lambda)(\lambda'_2 - \lambda)(\lambda'_3 - \lambda)$$

with

$$\begin{aligned}
\lambda_1' &= (\alpha'-1)(2\beta'+1) = \lambda_1 - p(2\alpha+2\beta-2p-1)\\
\lambda_2' &= \alpha'(2\beta'+3) = \lambda_2 - p(2\alpha+2\beta-2p+3)\\
\lambda_3' &= (2\alpha'-1)(\beta'+1) = \lambda_3 - p(2\alpha+2\beta-2p+1).
\end{aligned}$$

It is then easy to see (using the fact that $1 \le p \le \min\{\alpha-1,\beta\}$) that the only case when $\Delta_p(\lambda_j) = 0$ is when $p = 1$ and $\lambda_2' = \lambda_1$:

$$\lambda_2' - \lambda_1 = (1-p)(2\alpha+2\beta-2p+1).$$

The case $p = 1$, $\lambda = \lambda_1 = (\alpha-1)(2\beta+1)$ leads to

$$\begin{aligned}
-2(\alpha-1)a_1 - (2\beta-1)b_1 &= 2\beta(\alpha-1)\\
-(\alpha-1)a_1 - (\alpha+\beta-2)b_1 - \beta c_1 &= 2\beta(\alpha-1)\\
-(2\alpha-3)b_1 - 2\beta c_1 &= 2\beta(\alpha-1)
\end{aligned}$$

where the compatibility condition is satisfied, hence giving a one parameter family of eigenvectors.

Finally, it remains to study the cases when the limiting equations cannot be solved, i.e. the two cases

i) when $\alpha \le \beta$, $\lambda = 0$ (i.e. $\alpha = 0$, or 1), $p = \alpha$; the case $\alpha = 0$, $p = 0$, $\lambda = 0$ gives a known eigenvector, while $\alpha = p = 1$, $\lambda = 0$ gives $a_0 = b_0 = c_0 = 1$ and the equation for a_1 gives $0.a_1 = -2(a_0 - b_0) = 0$, hence the compatibility condition is satisfied.

ii) When $\beta \le \alpha-2$, $\lambda = \alpha-\beta-1$, $p = \beta+1$. The only possibility is $\lambda_1 = \alpha-\beta-1$ which happens if $\beta = 0$. Then $p = 1$, and we need to solve $c_1 = c_1 - 2(c_0 - b_0)$ where $a_0 = b_0 = c_0 = 1$. Hence the compatibility condition is satisfied. This ends the study in the first invariant subspace.

In conclusion, all the eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}'_{\alpha,\beta,\gamma}$ and thus in $\mathcal{H}'_k$ are non negative integers.

Step 4. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}''_k$. We also split $\mathcal{H}''_k$ into subspaces invariant by $\mathcal{A}_{L^*}\mathcal{A}_L$.

Step 4.1. Splitting of $\mathcal{H}''_k$. For $\alpha+\beta+2\gamma+1 = k$, let us denote $\mathcal{E}''_{\alpha,\beta,\gamma}$ the subspace of $\mathcal{H}''_k$ gathering the polynomials Φ of the form

$$\begin{aligned}
\phi_1 &= \sum_p a_p \psi_{\alpha-p,\beta-p,\gamma+p}\\
\phi_2 &= \sum_p b_p \phi_{\alpha-p,\beta-p+1,\gamma+p}\\
\phi_3 &= \sum_p c_p \psi_{\alpha-p-1,\beta-p+1,\gamma+p}
\end{aligned}$$

where

$$\begin{aligned}
\text{for } \alpha &\le \beta, \quad 0 \le p \le \alpha, \quad c_\alpha = 0\\
\text{for } \beta &\le \alpha-1, \quad 0 \le p \le \beta+1, \quad a_{\beta+1} = 0, \text{ and } c_{\beta+1} = 0 \text{ if } \alpha = \beta+1.
\end{aligned}$$

The following identities

$$\begin{array}{rcl}
\mathcal{D}\phi_1 - \phi_2 &=& \sum\{(2\alpha-2p+1)a_p - b_p\}\phi_{\alpha-p,\beta-p+1,\gamma+p} \\
&& \qquad -2(\alpha-p)a_p\phi_{\alpha-p-1,\beta-p,\gamma+p+1} \\
\mathcal{D}\phi_2 - \phi_3 &=& \sum\{(\alpha-p)b_p - c_p\}\psi_{\alpha-p-1,\beta-p+1,\gamma+p} \\
\mathcal{D}^*(\mathcal{D}\phi_1 - \phi_2) &=& \sum(\beta-p+1)\{(2\alpha-2p+1)a_p - b_p\}\psi_{\alpha-p,\beta-p,\gamma+p} + \\
&& \qquad -2(\alpha-p)(\beta-p)a_p\psi_{\alpha-p-1,\beta-p-1,\gamma+p+1} \\
\mathcal{D}^*(\mathcal{D}\phi_2 - \phi_3) &=& \sum(2\beta-2p+3)\{(\alpha-p)b_p - c_p\}\phi_{\alpha-p,\beta-p+1,\gamma+p} + \\
&& \qquad -2(\beta-p+1)\{(\alpha-p)b_p - c_p\}\phi_{\alpha-p-1,\beta-p,\gamma+p+1} \\
\mathcal{D}^*\mathcal{D}\phi_3 &=& \sum(2\alpha-2p-1)(\beta-p+2)c_p\psi_{\alpha-p-1,\beta-p+1,\gamma+p} + \\
&& \qquad -2(\alpha-p-1)(\beta-p+1)c_p\psi_{\alpha-p-2,\beta-p,\gamma+p+1}
\end{array}$$

ensure that subspace $\mathcal{E}''_{\alpha,\beta,\gamma}$ is invariant under $\mathcal{A}_{L^*}\mathcal{A}_L$.

Moreover, the two dimensional subspace $\mathcal{P}''_k = \mathrm{span}\{\Phi''_k, \Psi''_k\}$ where $\Phi''_k = (0, \phi_{k,0,0}, 0)$ and $\Psi''_k = (0, 0, \phi_{k-1,0,0})$ is stable by $\mathcal{A}_{L^*}\mathcal{A}_L$ since

$$\mathcal{A}_{L^*}\mathcal{A}_L\Phi''_k = (k+1)\Phi''_k - k\Psi''_k, \qquad \text{and} \qquad \mathcal{A}_{L^*}\mathcal{A}_L\Phi''_k = -\Phi''_k + 2k\Psi''_k.$$

Then, since $\Phi = (\psi_{\alpha,\beta,\gamma}, 0, 0)$, $\Phi = (0, \phi_{\alpha,\beta+1,\gamma}, 0)$, $\Phi = (0, \phi_{\alpha-1,\beta,\gamma+1}, 0)$, $\Phi = (0, 0, \psi_{\alpha-1,\beta+1,\gamma})$ and $\Phi = (0, 0, \psi_{\alpha-2,\beta,\gamma+1})$ belong to $\mathcal{E}''_{\alpha,\beta,\gamma}$ respectively for $\alpha \geq 0$, $\alpha \geq 0$, $\alpha \geq 1$, $\alpha \geq 1$ and $\alpha \geq 2$, we have the splitting of $\mathcal{H}''_k$ into the non direct sum

$$\mathcal{H}''_k = \mathcal{P}''_k + \sum_{\alpha+\beta+2\gamma=k} \mathcal{E}''_{\alpha,\beta,\gamma} \qquad \text{with } \mathcal{P}''_k = \mathrm{span}\{\Phi''_k, \Psi''_k\}.$$

Hence, the spectrum $\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_\mathrm{k}})$ *of the operator* $\mathcal{A}_{L^*}\mathcal{A}_L$ *in* $\mathcal{H}_k$ *is given by the union with possibly many overlaps*

$$\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_\mathrm{k}}) = \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{P}''_\mathrm{k}}) \;\cup \bigcup_{\alpha+\beta+2\gamma=\mathrm{k}} \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{E}'_{\alpha,\beta,\gamma}}).$$

Step 4.2. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{P}''_k$. In the basis $\{\Phi''_k, \Psi''_k\}$ the matrix of $\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{P}''_k}$ reads

$$\begin{pmatrix} k+1 & -k \\ -1 & 2k \end{pmatrix}.$$

Hence, the spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ *in* $\mathcal{P}''_k$ *is given by*

$$\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{P}''_\mathrm{k}}) = \{2\mathrm{k}+1, \mathrm{k}\}.$$

Step 4.3. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}''_{\alpha,\beta,\gamma}$. The spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$, for $\Phi \in \mathcal{E}''_{\alpha,\beta,\gamma}$ can be written as a hierarchy of systems of equation $(38)_p$ where for $p = 0$ we have

$$\begin{array}{rcl}
(\beta+1)\{(2\alpha+1)a_0 - b_0\} &=& \lambda a_0 \\
(2\beta+3)(\alpha b_0 - c_0) + b_0 - (2\alpha+1)a_0 &=& \lambda b_0, \\
(2\alpha-1)(\beta+2)c_0 + c_0 - \alpha b_0 &=& \lambda c_0,
\end{array} \qquad (38)_0$$

for $1 \leq p \leq \min\{\alpha, \beta+1\}$

$$\begin{array}{rcl}
\lambda a_p &=& (\beta-p+1)\{(2\alpha-2p+1)a_p - b_p\} - 2(\alpha-p+1)(\beta-p+1)a_{p-1}, \\
\lambda b_p &=& (2\beta-2p+3)\{(\alpha-p)b_p - c_p\} - (2\alpha-2p+1)a_p + b_p + \\
&& -2(\beta-p+2)\{(\alpha-p+1)b_{p-1} - c_{p-1}\} + 2(\alpha-p+1)a_{p-1}, \\
\lambda c_p &=& (2\alpha-2p-1)(\beta-p+2)c_p - (\alpha-p)b_p + c_p + \\
&& -2(\alpha-p)(\beta-p+2)c_{p-1}.
\end{array} \qquad (38)_p$$

In particular, when $\alpha \leq \beta$ the last system of the hierarchy is reached for $p = \alpha$ ($c_\alpha = 0$) and it reads

$$\begin{array}{rcl}
\lambda a_\alpha &=& (\beta-\alpha+1)(a_\alpha - b_\alpha) - 2(\beta-\alpha+1)a_{\alpha-1}, \\
\lambda b_\alpha &=& -a_\alpha + b_\alpha - 2(\beta-\alpha+2)(b_{\alpha-1} - c_{\alpha-1}) + 2a_{\alpha-1}, \\
0 &=& 0.
\end{array}$$

This last system enables to compute $a_\alpha,\ b_\alpha$ if $\lambda \neq 0$ and $\lambda \neq \beta - \alpha + 2$.

When $\beta \leq \alpha - 1$, the last system of the hierarchy is reached for $p = \beta + 1$ ($a_{\beta+1} = 0$ and $c_{\beta+1} = 0$ if $\beta = \alpha - 1$) and it reads

$$\begin{array}{rcl}
0 &=& 0, \\
\lambda b_{\beta+1} &=& (\alpha-\beta)b_{\beta+1} - c_{\beta+1} + 2\{(\alpha-\beta)(a_\beta - b_\beta) + c_\beta\}, \\
\lambda c_{\beta+1} &=& (\alpha-\beta-1)\{2c_{\beta+1} - b_{\beta+1} - 2c_\beta\}.
\end{array}$$

This last system enables to compute $b_\alpha,\ c_\alpha$ if $\lambda \neq 1$, when $\beta = \alpha - 1$ and if $\lambda \neq \alpha - \beta - 1$ and $\lambda \neq 2\alpha - 2\beta - 1$ when $\beta \leq \alpha - 2$.

The system for $p = 0$ gives the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ where

$$\begin{array}{ll}
\lambda_1 = \alpha(2\beta+3) & a_0 = \beta+1,\ b_0 = \beta-\alpha+1,\ c_0 = -\alpha, \\
\lambda_2 = (\beta+1)(2\alpha-1) & a_0 = 1,\ b_0 = 2,\ c_0 = 1, \\
\lambda_3 = (\beta+2)(2\alpha+1) & a_0 = (\beta+1)(2\beta+3),\ b_0 = -(2\alpha+1)(2\beta+3),\ c_0 = \alpha(2\alpha+1).
\end{array}$$

Notice that $\lambda_1 = 0$ for $\alpha = 0$, which corresponds to a already known eigenvector in the kernel of $\mathcal{A}_L$. The coefficients a_p, b_p, c_p can be computed by induction provided that for $\lambda = \lambda_1$ or λ_2 or λ_3 the determinant

$$\Delta_p(\lambda) = (\lambda_1' - \lambda)(\lambda_2' - \lambda)(\lambda_3' - \lambda)$$

does not cancel, where

$$\begin{array}{rcl}
\lambda_1' &=& \lambda_1 - p(2\alpha+2\beta-2p+3), \\
\lambda_2' &=& \lambda_2 - p(2\alpha+2\beta-2p+1), \\
\lambda_3' &=& \lambda_3 - p(2\alpha+2\beta-2p+5).
\end{array}$$

Using the fact that $1 \leq p \leq \min\{\alpha, \beta+1\}$, we can see that the only problem comes when $\lambda_3' = \lambda_2$:

$$\lambda_3' - \lambda_2 = (1-p)(2\alpha+2\beta+3-2p)$$

which occurs when $p = 1$. This case $p = 1$, $\lambda = (\beta+1)(2\alpha-1)$ gives the system

$$\begin{array}{rcl}
0 &=& -(2\alpha-1)a_1 - \beta b_1 - 2\alpha\beta a_0 \\
0 &=& -(2\alpha-1)a_1 + (1-\alpha-\beta)b_1 - (2\beta+1)c_1 - 2(\beta+1)(\alpha b_0 - c_0) + 2\alpha a_0 \\
0 &=& (1-\alpha)b_1 - (2\beta+1)c_1 - 2(\alpha-1)(\beta+1)c_0
\end{array}$$

where the compatibility condition is satisfied with the values we found for a_0, b_0, c_0 ($a_0 - b_0 + c_0 = 0$).

Finally, it then remains to study the last equation of the hierarchy:

i) when $\alpha \leq \beta$, $p = \alpha$, $\lambda = 0$ (i.e. $\alpha = p = 0$ leading to the know eigenvector in the kernel) or $\lambda = \beta - \alpha + 2$, i.e. $\lambda = \lambda_2$, $\alpha = 1 = p$ where the compatibility condition is satisfied due to $a_0 - b_0 + c_0 = 0$.

ii) When $\beta \leq \alpha - 1$, $p = \beta + 1$. Then for $\beta = \alpha - 1$, $\lambda_2 = 1$ (the bad case) for $\alpha = 1$ and this is again the case seen above. For $\beta \leq \alpha - 2$, the bad cases are when $\lambda_j = \alpha - \beta - 1$ or $2\alpha - 2\beta - 1$, i.e. $\lambda_2 = 2\alpha - 2\beta - 1$ for $\beta = 0$. We are again in the case $p = 1$ (notice that $a_1 = 0$) :

$$\begin{array}{rcl} 0 & = & (1-\alpha)b_1 - c_1 - 2(\alpha b_0 - c_0) + 2\alpha a_0 \\ 0 & = & (1-\alpha)b_1 - c_1 - 2(\alpha - 1)c_0 \end{array}$$

which is admits solutions since $a_0 - b_0 + c_0 = 0$.

In conclusion, all the eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}''_{\alpha,\beta,\gamma}$ and thus in $\mathcal{H}''_k$ are non negative integers. Gathering the results of step 3 and 4 we finally conclude that for every $k \geq 2$ all non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$ are positive integers. Hence, for every $k \geq 2$,

$$a_k := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

Lemma 2.22 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for $L = \mathbf{0^2 i\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$)
For $L = \mathbf{0^2 i\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ and for every $k \geq 2$, we have $\min\limits_{\lambda \in \Sigma_k(L)\setminus\{0\}} \{\lambda\} \geq \min\{1, \omega^2\}$ and thus

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq \max\{1, \omega^{-1}\}.$$

Proof. Since the real Jordan matrix $\mathbf{0^2 i\omega}|_{\mathbb{R}}$ is conjugated to the complex Jordan matrix $\mathbf{0^2 i\omega}|_{\mathbb{C}}$ via the unitary map

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix},$$

Remark 2.4 ensures that $a_k(\mathbf{0^2 i\omega}|_{\mathbb{C}}) = a_k(\mathbf{0^2 i\omega}|_{\mathbb{R}})$ for every $k \geq 2$. Thus, we make only the proof for $L = \mathbf{0^2 i\omega}|_{\mathbb{C}}$. Here again, we intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$. We proceed in several steps.

Step 1. Splitting of the homological operators. We are in dimension 4, with $X = (A, B, C, \widetilde{C})$ and

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega & 0 \\ 0 & 0 & 0 & -i\omega \end{pmatrix}.$$

In this case we have $\Phi = (\phi_A, \phi_B, \phi_C, \phi_{\widetilde{C}})$ and

$$\mathcal{A}_{L}\Phi = \begin{pmatrix} B\dfrac{\partial \phi_A}{\partial A} + i\omega C\dfrac{\partial \phi_A}{\partial C} - i\omega \widetilde{C}\dfrac{\partial \phi_A}{\partial \widetilde{C}} - \phi_B \\ B\dfrac{\partial \phi_B}{\partial A} + i\omega C\dfrac{\partial \phi_B}{\partial C} - i\omega \widetilde{C}\dfrac{\partial \phi_B}{\partial \widetilde{C}} \\ B\dfrac{\partial \phi_C}{\partial A} + i\omega C\dfrac{\partial \phi_C}{\partial C} - i\omega \widetilde{C}\dfrac{\partial \phi_C}{\partial \widetilde{C}} - i\omega\phi_C \\ B\dfrac{\partial \phi_{\widetilde{C}}}{\partial A} + i\omega C\dfrac{\partial \phi_{\widetilde{C}}}{\partial C} - i\omega \widetilde{C}\dfrac{\partial \phi_{\widetilde{C}}}{\partial \widetilde{C}} + i\omega\phi_{\widetilde{C}} \end{pmatrix},$$

$$\mathcal{A}_{L^*}\Psi = \begin{pmatrix} A\dfrac{\partial \psi_A}{\partial B} - i\omega C\dfrac{\partial \psi_A}{\partial C} + i\omega \widetilde{C}\dfrac{\partial \psi_A}{\partial \widetilde{C}} \\ A\dfrac{\partial \psi_B}{\partial B} - i\omega C\dfrac{\partial \psi_B}{\partial C} + i\omega \widetilde{C}\dfrac{\partial \psi_B}{\partial \widetilde{C}} - \psi_A \\ A\dfrac{\partial \psi_C}{\partial B} - i\omega C\dfrac{\partial \psi_C}{\partial C} + i\omega \widetilde{C}\dfrac{\partial \psi_C}{\partial \widetilde{C}} + i\omega\psi_C \\ A\dfrac{\partial \psi_{\widetilde{C}}}{\partial B} - i\omega C\dfrac{\partial \psi_{\widetilde{C}}}{\partial C} + i\omega \widetilde{C}\dfrac{\partial \psi_{\widetilde{C}}}{\partial \widetilde{C}} - i\omega\psi_{\widetilde{C}} \end{pmatrix},$$

$$\ker \mathcal{A}_{L} = \begin{cases} (B^\beta (C\widetilde{C})^\gamma, 0, 0, 0), & \beta + 2\gamma = k, \\ (AB^\beta (C\widetilde{C})^\gamma, B^{\beta+1}(C\widetilde{C})^\gamma, 0, 0), & \beta + 1 + 2\gamma = k, \\ (0, 0, B^\beta C (C\widetilde{C})^\gamma, B^\beta \widetilde{C}(C\widetilde{C})^\gamma), & \beta + 1 + 2\gamma = k. \end{cases}$$

Let us introduce some notations which simplify the writing of $\mathcal{A}_{L^*}\mathcal{A}_{L}$:

$$\mathcal{L}\Phi = \begin{cases} AB\dfrac{\partial^2 \phi_A}{\partial A \partial B} + A\dfrac{\partial \phi_A}{\partial A} - A\dfrac{\partial \phi_B}{\partial B} \\ AB\dfrac{\partial^2 \phi_B}{\partial A \partial B} + A\dfrac{\partial \phi_B}{\partial A} - B\dfrac{\partial \phi_A}{\partial A} + \phi_B \\ AB\dfrac{\partial^2 \phi_C}{\partial A \partial B} + A\dfrac{\partial \phi_C}{\partial A} \\ AB\dfrac{\partial^2 \phi_{\widetilde{C}}}{\partial A \partial B} + A\dfrac{\partial \phi_{\widetilde{C}}}{\partial A} \end{cases}$$

$$S = A\left\{C\frac{\partial^2}{\partial B \partial C} - \widetilde{C}\frac{\partial^2}{\partial B \partial \widetilde{C}}\right\} - B\left\{C\frac{\partial^2}{\partial A \partial C} - \widetilde{C}\frac{\partial^2}{\partial A \partial \widetilde{C}}\right\}$$

$$P = C^2\frac{\partial^2}{\partial C^2} + \widetilde{C}^2\frac{\partial^2}{\partial \widetilde{C}^2} + C\frac{\partial}{\partial C} + \widetilde{C}\frac{\partial}{\partial \widetilde{C}} - 2C\widetilde{C}\frac{\partial^2}{\partial C \partial \widetilde{C}}$$

then we can write $\mathcal{A}_{L^*}\mathcal{A}_{L}$ as

$$\begin{aligned} \mathcal{A}_{L^*}\mathcal{A}_{L}\Phi &= \mathcal{L}\Phi + i\omega S\Phi + \omega^2 P\Phi + i\omega\left\{C\frac{\partial}{\partial C} - \widetilde{C}\frac{\partial}{\partial \widetilde{C}}\right\}\begin{pmatrix} \phi_B \\ -\phi_A \\ 2i\omega\phi_C \\ -2i\omega\phi_{\widetilde{C}} \end{pmatrix} \\ &+ \left\{B\frac{\partial}{\partial A} - A\frac{\partial}{\partial B}\right\}\begin{pmatrix} 0 \\ 0 \\ i\omega\phi_C \\ -i\omega\phi_{\widetilde{C}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \omega^2\phi_C \\ \omega^2\phi_{\widetilde{C}} \end{pmatrix}. \end{aligned}$$

Step 2. Splitting of $\mathcal{H}_k$. We observe that the subspace $\mathcal{E}_\eta$ of linear combinations of monomials $A^\alpha B^\beta C^\gamma \widetilde{C}^\delta$ where the powers of C and $\widetilde{C}$ satisfy $\eta := \gamma - \delta$ fixed is invariant under $\mathcal{A}_{_L}|_{\mathcal{H}_k}$ and $\mathcal{A}_{_{L^*}}|_{\mathcal{H}_k}$. In $\mathcal{E}_\eta$, we have

$$\begin{aligned} S\Phi &= \eta\left\{A\frac{\partial}{\partial B} - B\frac{\partial}{\partial A}\right\}\Phi \\ P\Phi &= \eta^2\Phi. \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{_{L^*}}\mathcal{A}_{_L}\Phi &= \mathcal{L}\Phi + i\omega\left\{A\frac{\partial}{\partial B} - B\frac{\partial}{\partial A}\right\}\begin{pmatrix} \eta\ \phi_A \\ \eta\ \phi_B \\ (\eta-1)\phi_C \\ (\eta+1)\phi_{\widetilde{C}} \end{pmatrix} + \\ &\quad + i\omega(\gamma-\delta)\begin{pmatrix} \phi_B \\ -\phi_A \\ 0 \\ 0 \end{pmatrix} + \omega^2\begin{pmatrix} \eta^2\ \phi_A \\ \eta^2\ \phi_B \\ (\eta-1)^2\phi_C \\ (\eta+1)^2\phi_{\widetilde{C}} \end{pmatrix}. \end{aligned}$$

Moreover the three subspaces

$$\begin{aligned} \mathcal{H}_{_{AB}} &= \{\Phi \in \mathcal{H}_k/\Phi = (\phi_A, \phi_B, 0, 0)\}, \\ \mathcal{H}_{_C} &= \{\Phi \in \mathcal{H}_k/\Phi = (0, 0, \phi_C, 0)\}, \\ \mathcal{H}_{_{\widetilde{C}}} &= \{\Phi \in \mathcal{H}_k/\Phi = (0, 0, 0, \phi_{\widetilde{C}})\}, \end{aligned}$$

are also invariant under $\mathcal{A}_{_L}|_{\mathcal{H}_k}$ and $\mathcal{A}_{_{L^*}}|_{\mathcal{H}_k}$ and

$$\mathcal{H}_k = \bigoplus_{-k\le\eta\le k} (\mathcal{E}_\eta \cap \mathcal{H}_{_{AB}}) \oplus (\mathcal{E}_\eta \cap \mathcal{H}_{_C}) \oplus \left(\mathcal{E}_\eta \cap \mathcal{H}_{_{\widetilde{C}}}\right).$$

Hence, the spectrum of $\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}$ in $\mathcal{H}_k$ is the union of its spectrum in each subspaces of the direct sum.

Step 3. Spectrum of $\boldsymbol{\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}}$ in $\boldsymbol{\mathcal{E}_\eta \cap \mathcal{H}_{_{AB}}}$. For computing the spectrum of $\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}$ in $\mathcal{H}_{_{AB}} \cap \mathcal{E}_\eta$, we identify the operators $\mathcal{L}|_{\mathcal{H}_{AB}}$ and $\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}|_{\mathcal{H}_{AB}}$ with their restriction $\mathcal{L}_{_{AB}}$ and $(\mathcal{A}_{_{L^*}}\mathcal{A}_{_L})_{_{AB}}$ to the two first components. With this identification the spectral equation

$$\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}\Phi = \lambda\Phi \tag{39}$$

in $\mathcal{H}_{_{AB}} \cap \mathcal{E}_\eta$ reads

$$\mathcal{L}_{_{AB}}\Phi + i\omega\eta\mathcal{B}\Phi = (\lambda - \omega^2\eta^2)\Phi \tag{40}$$

where

$$\mathcal{B}\Phi = \left\{A\frac{\partial}{\partial B} - B\frac{\partial}{\partial A}\right\}\begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix} + \begin{pmatrix} \phi_B \\ -\phi_A \end{pmatrix}.$$

Step 3.1. Spectrum of $\boldsymbol{\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}}$ in $\boldsymbol{\mathcal{E}_0 \cap \mathcal{H}_{_{AB}}}$. Observing that for $\eta = 0$ the spectral equation (39) reads $\mathcal{L}_{_{AB}}\Phi = \lambda\Phi$ and that $\mathcal{L}_{_{AB}}$ is the same operator as the one involved in the 0^2 singularity studied in Lemma 2.20 (see (36)), we get that all eigenvalues of $\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}$ in $\mathcal{E}_0 \cap \mathcal{H}_{_{AB}}$ are positive integers and thus

$$\inf_{\substack{\lambda\in\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{E}_0\cap\mathcal{H}_{\mathrm{AB}}})\\ \lambda\neq 0}} \{\lambda\} \ge 1.$$

Step 3.2. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}_\eta \cap \mathcal{H}_{AB}$ for $\eta \neq 0$. In this case, observing that $\mathcal{L}_{AB}$ is a positive self adjoint operator and that for any $\Phi = (\phi_A, \phi_B)$ we have $\langle \mathcal{B}\Phi, \Phi\rangle = 0$, it then results from (40) that

$$\lambda \geq \omega^2\eta^2 \geq \omega^2.$$

since $\eta = \gamma - \delta \neq 0$ is an integer. Hence, for every η,

$$\inf_{\substack{\lambda\in \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{E}_\eta\cap\mathcal{H}_{\mathrm{AB}}})\\ \lambda\neq 0}} \{\lambda\} \geq \min\{\omega^2, 1\}.$$

Step 4. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}_\eta \cap \mathcal{H}_C$. In $\mathcal{H}_C \cap \mathcal{E}_\eta$, the spectral equation (39) reads

$$\mathrm{i}\omega \left\{A\frac{\partial}{\partial B} - B\frac{\partial}{\partial A}\right\}(\eta-1)\Phi_C + AB\frac{\partial^2\phi_C}{\partial A\partial B} + A\frac{\partial \phi_C}{\partial A} = (\lambda - (\eta-1)^2)\phi_C. \quad (41)$$

Step 4.1. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}_1 \cap \mathcal{H}_C$. In $\mathcal{H}_C \cap \mathcal{E}_1$, the spectral equation (39) reduces to

$$AB\frac{\partial^2\phi_C}{\partial A\partial B} + A\frac{\partial \phi_C}{\partial A} = \lambda\phi_C$$

which gives $\lambda = \alpha(\beta+1)$, with $\alpha+\beta+2\gamma-1 = k$, with eigenvectors belonging to the canonical basis. This leads again to

$$\inf_{\substack{\lambda\in \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{E}_1\cap\mathcal{H}_{\mathrm{C}}})\\ \lambda\neq 0}} \{\lambda\} \geq 1.$$

Step 4.1. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}_\eta \cap \mathcal{H}_C$ with $\eta \neq 1$. For $\eta - 1 \neq 0$, we observe again that $\left\{A\frac{\partial}{\partial B} - B\frac{\partial}{\partial A}\right\}\phi_C$ is orthogonal to ϕ_C and that $\Phi_C \mapsto AB\frac{\partial^2\phi_C}{\partial A\partial B} + A\frac{\partial\phi_C}{\partial A} = \lambda\phi_C$ is self adjoint and positive. Hence, we deduce from (41) that

$$\lambda \geq \omega^2(\eta-1)^2 \geq \omega^2$$

since $\eta = \gamma - \delta \neq 1$ is an integer. Hence, for every η,

$$\inf_{\substack{\lambda\in \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{E}_\eta\cap\mathcal{H}_{\mathrm{C}}})\\ \lambda\neq 0}} \{\lambda\} \geq \min\{\omega^2, 1\}.$$

Proceeding similarly, we get the same result for $\mathcal{H}_{\widetilde{C}}$. Collecting all the results, this shows that for every $k \geq 2$, we have $\min_{\lambda\in\Sigma_k(L)\backslash\{0\}}\{\lambda\} \geq \min\{1, \omega^2\}$ and thus

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|^{-1}_{\mathcal{H}_k}|||_2 \leq \max\{1, \omega^{-1}\}.$$

$\square$

Lemma 2.23 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for $L = \mathbf{0^2.i\omega_1.\cdots.i\omega_q}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$)

(a) *For $L = \mathbf{0^2.i\omega_1.\cdots.i\omega_q}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ where $\omega := (\omega_1, \cdots, \omega_q, -\omega_1, \cdots, -\omega_q) \in \mathbb{R}^{2q}$ is γ, τ-homologically diophantine, we have for every $k \geq 2$, $\min_{\lambda \in \Sigma_k(L)\setminus\{0\}} \{\lambda\} \geq \min\{1, \frac{\gamma^2}{k^{2\tau}}\}$ and thus*

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq \max\{1, \gamma^{-1}k^\tau\} \leq ak^\tau$$

where $a = \max\{2^{-\tau}, \gamma^{-1}\}$.

(b) *If ω is γ, K-homologically nonresonant, then there exist $a' > 0$ such that for every $k \geq 2$,*

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq a'.$$

Proof. As for $L = \mathbf{0^2}\boldsymbol{i\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$, since the real Jordan matrix $L = \mathbf{0^2.i\omega_1.\cdots.i\omega_q}|_{\mathbb{R}}$ is conjugated to the complex Jordan matrix $L = \mathbf{0^2.i\omega_1.\cdots.i\omega_q}|_{\mathbb{C}}$ via a unitary map, Remark 2.4 ensures that $a_k(\mathbf{0^2.i\omega_1.\cdots.i\omega_q}|_{\mathbb{R}}) = a_k(\mathbf{0^2.i\omega_1.\cdots.i\omega_q}|_{\mathbb{C}})$ for every $k \geq 2$. Thus, we make only the proof for $L = \mathbf{0^2.i\omega_1.\cdots.i\omega_q}|_{\mathbb{C}}$. Here again, we intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$. We make the proof only when ω is γ, τ-homologically diophantine. When ω is γ, K-homologically nonresonant, the proof is very similar and the details are left to the reader.

So, we are in dimension $2q+2$, with $X = (A, B, C_1, ...C_q, \widetilde{C}_1, ...\widetilde{C}_q)$ and

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & 0 & . & & . \\ 0 & 0 & \mathrm{i}\omega_1 & 0 & . & & . \\ 0 & 0 & 0 & \mathrm{i}\omega_2 & 0 & & . \\ . & & & 0 & . & 0 & 0 \\ . & & & & 0 & -\mathrm{i}\omega_{q-1} & 0 \\ 0 & . & . & . & 0 & 0 & -\mathrm{i}\omega_q \end{pmatrix}.$$

The homological operator $\mathcal{A}_L$ reads

$$\mathcal{A}_L\Phi = \begin{pmatrix} B\dfrac{\partial\phi_A}{\partial A} - \phi_B + \sum i\omega_j(C_j\dfrac{\partial\phi_A}{\partial C_j} - \widetilde{C}_j\dfrac{\partial\phi_A}{\partial\widetilde{C}_j}) \\ B\dfrac{\partial\phi_B}{\partial A} + \sum i\omega_j(C_j\dfrac{\partial\phi_B}{\partial C_j} - \widetilde{C}_j\dfrac{\partial\phi_B}{\partial\widetilde{C}_j}) \\ B\dfrac{\partial\phi_{C_1}}{\partial A} + \sum i\omega_j(C_j\dfrac{\partial\phi_{C_1}}{\partial C_j} - \widetilde{C}_j\dfrac{\partial\phi_{C_1}}{\partial\widetilde{C}_j}) - i\omega_1\phi_{C_1} \\ \vdots \\ B\dfrac{\partial\phi_{\widetilde{C}_q}}{\partial A} + \sum i\omega_j(C_j\dfrac{\partial\phi_{\widetilde{C}_q}}{\partial C_j} - \widetilde{C}_j\dfrac{\partial\phi_{\widetilde{C}_q}}{\partial\widetilde{C}_j}) + i\omega_q\phi_{\widetilde{C}_q} \end{pmatrix}$$

Let us introduce notations which simplify the writing of $\mathcal{A}_{L^*}\mathcal{A}_L$. We define operators $\mathcal{L}$, $\mathcal{S}, Q$ by

$$\mathcal{L}\Phi = \begin{pmatrix} AB\dfrac{\partial^2\phi_A}{\partial A\partial B} + A\dfrac{\partial\phi_A}{\partial A} - A\dfrac{\partial\phi_B}{\partial B} \\ AB\dfrac{\partial^2\phi_B}{\partial A\partial B} + A\dfrac{\partial\phi_B}{\partial A} - B\dfrac{\partial\phi_A}{\partial A} + \phi_B \\ AB\dfrac{\partial^2\phi_{C_1}}{\partial A\partial B} + A\dfrac{\partial\phi_{C_1}}{\partial A} \\ \vdots \\ AB\dfrac{\partial^2\phi_{\widetilde{C}_q}}{\partial A\partial B} + A\dfrac{\partial\phi_{\widetilde{C}_q}}{\partial A} \end{pmatrix}$$

$$\mathcal{S} = \sum_{1\le j\le n}\omega_j(C_j\frac{\partial}{\partial C_j} - \widetilde{C}_j\frac{\partial}{\partial \widetilde{C}_j}), \qquad Q = A\frac{\partial}{\partial B} - B\frac{\partial}{\partial A}$$

then we have

$$\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \mathcal{L}\Phi + iQ\mathcal{S}\Phi + \mathcal{S}^2\Phi - Q\begin{pmatrix} 0 \\ 0 \\ i\omega_1\phi_{C_1} \\ \vdots \\ -i\omega_q\phi_{\widetilde{C}_q}\end{pmatrix} + i\mathcal{S}\begin{pmatrix} \phi_B \\ -\phi_A \\ 2i\omega_1\phi_{C_1} \\ \vdots \\ -2i\omega_q\phi_{\widetilde{C}_q}\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \omega_1^2\phi_{C_1} \\ \vdots \\ \omega_q^2\phi_{\widetilde{C}_q}\end{pmatrix}.$$

The kernel of $\mathcal{A}_L$, which is also the kernel of $\mathcal{A}_{L^*}\mathcal{A}_L$ is formed by the following vectors

$$\begin{array}{l} (B^\beta(C_1\widetilde{C}_1)^{\gamma_1}\cdots(C_q\widetilde{C}_q)^{\gamma_q}, 0, 0, \cdots, 0) \\ (AB^\beta(C_1\widetilde{C}_1)^{\gamma_1}\cdots(C_q\widetilde{C}_q)^{\gamma_q}, B^{\beta+1}(C_1\widetilde{C}_1)^{\gamma_1}\cdots(C_q\widetilde{C}_q)^{\gamma_q}, 0, \cdots, 0) \\ (0, 0, B^\beta C_1(C_1\widetilde{C}_1)^{\gamma_1}\cdots(C_q\widetilde{C}_q)^{\gamma_q}, 0, \cdots, B^\beta\widetilde{C}_1(C_1\widetilde{C}_1)^{\gamma_1}\cdots(C_q\widetilde{C}_q)^{\gamma_q}, 0, \cdots, 0) \\ \qquad\qquad\vdots \\ (0, 0, \cdots, 0, B^\beta C_q(C_1\widetilde{C}_1)^{\gamma_1}\cdots(C_q\widetilde{C}_q)^{\gamma_q}, 0, \cdots, 0, B^\beta\widetilde{C}_q(C_1\widetilde{C}_1)^{\gamma_1}, \cdots, (C_q\widetilde{C}_q)^{\gamma_q}). \end{array}$$

Let us proceed as for $L = \mathbf{0^2 i\boldsymbol{\omega}}|_{\mathbb{C}}$, and observe that here again, the subspaces

$$\begin{aligned} \mathcal{H}_{AB} &= \{\Phi\in\mathcal{H}_k/\Phi = (\phi_A, \phi_B, 0, \cdots, 0)\}, \\ \mathcal{H}_{C_j} &= \{\Phi\in\mathcal{H}_k/\Phi = (0, 0, \cdots, 0, \phi_{C_j}, 0, \cdots, 0)\}, \\ \mathcal{H}_{\widetilde{C}_j} &= \{\Phi\in\mathcal{H}_k/\Phi = (0, 0, \cdots, 0, \phi_{\widetilde{C}_j}, 0, \cdots, 0)\}, \end{aligned}$$

are invariant under $\mathcal{A}_L|_{\mathcal{H}_k}$ and $\mathcal{A}_{L^*}|_{\mathcal{H}_k}$, and that $\mathcal{H}_k = \mathcal{H}_{AB}\bigoplus_{1\le j\le q}\mathcal{H}_{C_j}\bigoplus_{1\le j\le q}\mathcal{H}_{\widetilde{C}_j}$.

For computing the spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_{AB}$, denote by $\mathcal{L}_{AB}$ the restriction of $\mathcal{L}$ to the two first components. We observe that $Q\begin{pmatrix}\phi_A\\ \phi_B\end{pmatrix}$ and $\begin{pmatrix}\phi_B\\ -\phi_A\end{pmatrix}$ are orthogonal to $\begin{pmatrix}\phi_A\\ \phi_B\end{pmatrix}$. Hence the method used for $L = \mathbf{0^2 i\boldsymbol{\omega}}|_{\mathbb{C}}$ applies and shows, using the reduced operator $\mathcal{L}_{AB} + \mathcal{S}^2$ that the eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_{AB}$ satisfy

$$\min_{\substack{\lambda\in\mathrm{spec}(\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}_{AB}}) \\ \lambda\neq 0}}\{\lambda\} \ \ge \min_{\substack{\sum\gamma_j+\delta_j\le k \\ 1\le j\le q}}\left\{1, \left(\sum(\gamma_j-\delta_j)\omega_j\right)^2\right\} \ge \min\{1, \frac{\gamma^2}{k^{2\nu}}\}.$$

Now in $\mathcal{H}_{C_j}$, the spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$ reads

$$AB\frac{\partial^2\phi_{C_j}}{\partial A\partial B} + A\frac{\partial\phi_{C_j}}{\partial A} + iQ(\mathcal{S}-\omega_j)\phi_{C_j} + (\mathcal{S}-\omega_j)^2\phi_{C_j} = \lambda\phi_{C_j}.$$

Looking at the eigenvalues of the selfadjoint operator $AB\frac{\partial^2}{\partial A\partial B}+A\frac{\partial}{\partial A}+(\mathcal{S}-\omega_j)^2$, and applying the same method as for $L=\mathbf{0^2 i\omega}$, we find in this subspace

$$\min_{\substack{\lambda\in\mathrm{spec}(\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}_{C_j}})\\ \lambda\neq 0}}\{\lambda\} \geq \min_{\substack{\sum\limits_{1\leq j\leq q}\gamma_j+\delta_j\leq k}}\left\{1,\left(\sum_{1\leq\ell\leq q}(\gamma_\ell-\delta_\ell)\omega_\ell\pm\omega_j\right)^2\right\}\geq \min\{1,\frac{\gamma^2}{k^{2\nu}}\}.$$

The same inequalities holds in $\mathcal{H}_{\widetilde{C}_j}$. Collecting all the results, this shows that for every $k\geq 2$, we have $\min\limits_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\}\geq\min\{1,\frac{\gamma^2}{k^{2\tau}}\}$ and thus

$$a_k(L):=|||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2\leq\max\{1,\gamma^{-1}k^\tau\}.$$

□

Lemma 2.24 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for $L=(\mathbf{i\omega})^2|_{\mathbb{R}\,\mathbf{or}\,\mathbb{C}}$)
For $L=(\mathbf{i\omega})^2|_{\mathbb{R}\,\mathbf{or}\,\mathbb{C}}$ and for every $k\geq 2$, we have $\min\limits_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\}\geq\min\{1,\omega^2\}$ *and thus*

$$a_k(L):=|||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2\leq\max\{1,\omega^{-1}\}.$$

Proof. As for the $(\boldsymbol{i\omega})^2|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$ singularity, since the real Jordan matrix $L=(\boldsymbol{i\omega})^2|_{\mathbb{R}}$ is conjugated to the complex Jordan matrix $L=(\boldsymbol{i\omega})^2|_{\mathbb{C}}$ via a unitary map, Remark 2.4 ensures that $a_k((\boldsymbol{i\omega})^2|_{\mathbb{R}})=a_k((\boldsymbol{i\omega})^2|_{\mathbb{C}})$ for every $k\geq 2$. Thus, we make only the proof for $L=(\boldsymbol{i\omega})^2|_{\mathbb{C}}$. Here again, we intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$. So we are in dimension 4, with $X=(A,B,\widetilde{A},\widetilde{B})$, $\Phi=(\phi_A,\phi_B,\phi_{\widetilde{A}},\phi_{\widetilde{B}})$ and

$$L=\begin{pmatrix} i\omega & 1 & 0 & 0\\ 0 & i\omega & 0 & 0\\ 0 & 0 & -i\omega & 1\\ 0 & 0 & 0 & -i\omega\end{pmatrix}.$$

Step 1. Splitting of the homological operators. Denoting $\Phi_{AB}=(\phi_A,\phi_B)$ and $\Phi_{\widetilde{A}\widetilde{B}}=(\phi_{\widetilde{A}},\phi_{\widetilde{B}})$, the homological operators read

$$\mathcal{A}_L\Phi=\begin{pmatrix} i\omega(S-I)\Phi_{AB}+\mathcal{B}\Phi_{AB}-\begin{pmatrix}\phi_B\\0\end{pmatrix}\\ \\ -i\omega(S-I)\Phi_{\widetilde{A}\widetilde{B}}+\mathcal{B}\Phi_{\widetilde{A}\widetilde{B}}-\begin{pmatrix}\phi_{\widetilde{B}}\\0\end{pmatrix}\end{pmatrix},$$

$$\mathcal{A}_{L^*}\Psi=\begin{pmatrix} -i\omega(S-I)\Psi_{AB}+\mathcal{B}^*\Psi_{AB}-\begin{pmatrix}0\\\psi_A\end{pmatrix}\\ \\ i\omega(S-I)\Psi_{\widetilde{A}\widetilde{B}}+\mathcal{B}^*\Psi_{\widetilde{A}\widetilde{B}}-\begin{pmatrix}0\\\psi_{\widetilde{A}}\end{pmatrix}\end{pmatrix}$$

where

$$\begin{aligned} S &= A\frac{\partial}{\partial A}-\widetilde{A}\frac{\partial}{\partial\widetilde{A}}+B\frac{\partial}{\partial B}-\widetilde{B}\frac{\partial}{\partial\widetilde{B}},\\ \mathcal{B} &= B\frac{\partial}{\partial A}+\widetilde{B}\frac{\partial}{\partial\widetilde{A}},\\ \mathcal{B}^* &= A\frac{\partial}{\partial B}+\widetilde{A}\frac{\partial}{\partial\widetilde{B}}.\end{aligned}$$

Step 2. Splitting of $\mathcal{H}_k$. Observe that the two supplementary subspaces

$$\mathcal{H}_{AB} = \{\Phi \in \mathcal{H}_k / \Phi = (\phi_A, \phi_B, 0, 0)\}, \qquad \mathcal{H}_{\widetilde{A}\widetilde{B}} = \{\Phi \in \mathcal{H}_k / \Phi = (0, 0, \phi_{\widetilde{A}}, \phi_{\widetilde{B}})\},$$

are both invariant by $\mathcal{A}_L$ and $\mathcal{A}_{L^*}$. Hence the spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$ is the union of its spectrum in each subspaces. So we begin with the computation of the spectrum of $\mathcal{A}_L\mathcal{A}_{L^*}$ in $\mathcal{H}_{AB}$. The computation in $\mathcal{H}_{\widetilde{A}\widetilde{B}}$ is totally similar since $\mathcal{A}_L\mathcal{A}_{L^*}|_{\mathcal{H}_{\widetilde{A}\widetilde{B}}}$ is equal to $\mathcal{A}_L\mathcal{A}_{L^*}|_{\mathcal{H}_{AB}}$ up to the change of the sign of ω.

Step 3. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_{AB}$. For computing the spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_{AB}$, we identify Φ and the operator $\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}_{AB}}$ with their restrictions Φ_{AB} and $(\mathcal{A}_{L^*}\mathcal{A}_L)_{AB}$ to the two first components. With this identification $\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}_{AB}}$ reads

$$\begin{aligned} \mathcal{A}_L\mathcal{A}_{L^*}|_{\mathcal{H}_{AB}}\Phi &= \omega^2(S-I)^2\Phi + \mathcal{B}^*\mathcal{B}\Phi + i\omega\{\mathcal{B}^*(S-I) - (S-I)\mathcal{B}\}\Phi + \\ &\quad + i\omega(S-I)\begin{pmatrix} \phi_B \\ -\phi_A \end{pmatrix} - \begin{pmatrix} \mathcal{B}^*\phi_B \\ \mathcal{B}\phi_A \end{pmatrix} + \begin{pmatrix} 0 \\ \phi_B \end{pmatrix}. \end{aligned}$$

Due to the selfadjointness of S, we observe that $\{\mathcal{B}^*(S-I)-(S-I)\mathcal{B}\}\Phi$ and $(S-I)\begin{pmatrix} \phi_B \\ -\phi_A \end{pmatrix}$ are orthogonal to Φ. Then, the minmax principle ensures that

$$\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_{\mathrm{AB}}}) = \mathrm{spec}(\mathcal{T}) \tag{42}$$

where $\mathcal{T} : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$ is the selfadjoint reduced operator

$$\mathcal{T}\Phi = \omega^2(S-I)^2\Phi + \mathcal{B}^*\mathcal{B}\Phi - \begin{pmatrix} \mathcal{B}^*\phi_B \\ \mathcal{B}\phi_A \end{pmatrix} + \begin{pmatrix} 0 \\ \phi_B \end{pmatrix}. \tag{43}$$

Moreover the kernel of $\mathcal{A}_L$ and $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_{AB}$ is formed with the vectors

$$\begin{aligned} &(B(B\widetilde{B})^\beta(A\widetilde{B} - B\widetilde{A})^\gamma, 0) \\ &(A(B\widetilde{B})^\beta(A\widetilde{B} - B\widetilde{A})^\gamma, B(B\widetilde{B})^\beta(A\widetilde{B} - B\widetilde{A})^\gamma) \end{aligned}$$

where we notice that all eigenvectors satisfy $(S-I)\Phi = 0$, and $(S-I)\mathcal{B}\Phi = 0$, and thus belong to the kernel of $\mathcal{T}$.

Step 4. Spectrum of $\mathcal{T}$. *The problem is then reduced to the computation of the minimum of nonzero eigenvalues of $\mathcal{T}$ in $\mathcal{H}_{AB}$.*

Step 4.1 Splitting of $\mathcal{H}_{AB}$. To compute the spectrum of $\mathcal{T}$, we use the non canonical basis of monomials given by

$$\begin{aligned} \phi_{\alpha,\beta_1,\beta_2,\gamma} &= A^\alpha B^{\beta_1}\widetilde{B}^{\beta_2}(A\widetilde{B} - B\widetilde{A})^\gamma, \\ \psi_{\alpha_1,\alpha_2,\beta,\gamma} &= A^{\alpha_1}\widetilde{A}^{\alpha_2}\widetilde{B}^\beta(A\widetilde{B} - B\widetilde{A})^\gamma, \\ \psi_{\alpha,0,\beta,\gamma} &\equiv \phi_{\alpha,0,\beta,\gamma}. \end{aligned}$$

Using the following properties for the derivation operators $S, \mathcal{B}, \mathcal{B}^*$

$$\begin{array}{rcl} S(A\widetilde{B} - B\widetilde{A}) &=& 0, \\ \mathcal{B}(A\widetilde{B} - B\widetilde{A}) &=& 0, \quad \mathcal{B}(B) = 0, \\ \mathcal{B}^*(A\widetilde{B} - B\widetilde{A}) &=& 0, \quad \mathcal{B}^*(A) = 0, \\ S(B) &=& B, \;\; S(A) = A, \\ S(\widetilde{B}) &=& -\widetilde{B}, \;\; S(\widetilde{A}) = -\widetilde{A}, \\ \mathcal{B}(A) &=& B, \;\; \mathcal{B}(\widetilde{A}) = \widetilde{B}, \\ \mathcal{B}^*(B) &=& A, \;\; \mathcal{B}^*(\widetilde{B}) = \widetilde{A}, \end{array}$$

we obtain

$$\begin{array}{rcl} \omega^2(S-I)^2\phi_{\alpha,\beta_1,\beta_2,\gamma} &=& \omega^2(\alpha+\beta_1-\beta_2-1)^2\phi_{\alpha,\beta_1,\beta_2,\gamma}, \\ \omega^2(S-I)^2\psi_{\alpha_1,\alpha_2,\beta,\gamma} &=& \omega^2(\alpha_1-\alpha_2-\beta-1)^2\psi_{\alpha_1,\alpha_2,\beta,\gamma}, \\ \mathcal{B}\phi_{\alpha,\beta_1,\beta_2,\gamma} &=& \alpha\phi_{\alpha-1,\beta_1+1,\beta_2,\gamma}, \\ \mathcal{B}\psi_{\alpha_1,\alpha_2,\beta,\gamma} &=& (\alpha_1+\alpha_2)\psi_{\alpha_1,\alpha_2-1,\beta+1,\gamma} - \alpha_1\psi_{\alpha_1-1,\alpha_2-1,\beta,\gamma+1}, \\ \mathcal{B}^*\phi_{\alpha,\beta_1,\beta_2,\gamma} &=& (\beta_1+\beta_2)\phi_{\alpha+1,\beta_1-1,\beta_2,\gamma} - \beta_2\phi_{\alpha,\beta_1-1,\beta_2-1,\gamma+1}, \\ \mathcal{B}^*\psi_{\alpha_1,\alpha_2,\beta,\gamma} &=& \beta\psi_{\alpha_1,\alpha_2+1,\beta-1,\gamma}, \end{array}$$

$$\begin{array}{rcl} \mathcal{B}^*\mathcal{B}\phi_{\alpha,\beta_1,\beta_2,\gamma} &=& \alpha(\beta_1+\beta_2+1)\phi_{\alpha,\beta_1,\beta_2,\gamma} - \alpha\beta_2\phi_{\alpha-1,\beta_1,\beta_2-1,\gamma+1}, \\ \mathcal{B}^*\mathcal{B}\psi_{\alpha_1,\alpha_2,\beta,\gamma} &=& (\alpha_1+\alpha_2)(\beta+1)\psi_{\alpha_1,\alpha_2,\beta,\gamma} - \alpha_1\beta\psi_{\alpha_1-1,\alpha_2,\beta-1,\gamma+1}. \end{array}$$

Let introduce the two supplementary subspaces

$$\begin{array}{rcl} \mathcal{H}'_{AB} &=& \Big\{(\Phi = (\phi_A, \phi_B)/\Phi_A \in \underset{\alpha+\beta_1+\beta_2+2\gamma=k}{\text{span}} \{\phi_{\alpha,\beta_1,\beta_2,\gamma}\},\ \Phi_B \in \underset{\substack{\alpha+\beta_1+\beta_2+2\gamma=k \\ \beta_1\geq 1}}{\text{span}} \{\phi_{\alpha,\beta_1,\beta_2,\gamma}\}\Big\} \\ \mathcal{H}''_{AB} &=& \Big\{(\Phi = (\phi_A, \phi_B)/\Phi_A \in \underset{\substack{\alpha+\beta_1+\beta_2+2\gamma=k \\ \alpha_2\geq 1}}{\text{span}} \{\psi_{\alpha_1,\alpha_2,\beta,\gamma}\},\ \Phi_B \in \underset{\alpha_1+\alpha_2+\beta+\gamma=k}{\text{span}} \{\psi_{\alpha_1,\alpha_2,\beta,\gamma}\}\Big\} \end{array}$$

and observe that $\mathcal{H}'_{AB}$ and $\mathcal{H}''_{AB}$ are both invariant under $\mathcal{T}$. *Hence, the spectrum of the operator $\mathcal{T}$ in $\mathcal{H}_{AB}$ is the union of its spectrum when restricted to $\mathcal{H}'_{AB}$ and to $\mathcal{H}''_{AB}$.*

Step 4.2. Spectrum of $\mathcal{T}$ in $\mathcal{H}'_{AB}$. We also split $\mathcal{H}'_{AB}$ into subspaces invariant by $\mathcal{T}$.

Step 4.2.1. Splitting of $\mathcal{H}'_{AB}$. First observe that for $\alpha+\beta_1+\beta_2+2\gamma = k$, the subspace $\mathcal{E}'_{\alpha,\beta_1,\beta_2,\gamma}$ of $\mathcal{H}'_{AB}$ gathering the polynomials Φ of the form

$$\begin{array}{rcl} \phi_A &=& \sum_p a_p\phi_{\alpha-p,\beta_1,\beta_2-p,\gamma+p} \\ \phi_B &=& \sum_p b_p\phi_{\alpha-p-1,\beta_1+1,\beta_2-p,\gamma+p} \end{array} \tag{44}$$

where

$$\begin{array}{ll} \text{for } \beta_2 \leq \alpha - 1, & 0 \leq p \leq \beta_2, \\ \text{for } \alpha \leq \beta_2 & 0 \leq p \leq \alpha, \quad \text{with } b_\alpha = 0. \end{array}$$

is invariant under the operator $\mathcal{T}$. Indeed we have

$$\mathcal{B}\phi_A - \phi_B = \sum_p \{(\alpha-p)a_p - b_p\}\phi_{\alpha-p-1,\beta_1+1,\beta_2-p,\gamma+p},$$

$$\begin{aligned}\mathcal{B}^*(\mathcal{B}\phi_A - \phi_B) &= \sum_p \{(\alpha - p)a_p - b_p\}(\beta_1 + \beta_2 - p + 1)\phi_{\alpha-p,\beta_1,\beta_2-p,\gamma+p} + \\ &\quad - \sum_p \{(\alpha - p)a_p - b_p\}(\beta_2 - p)\phi_{\alpha-p-1,\beta_1,\beta_2-p-1,\gamma+p+1},\end{aligned}$$

$$\begin{aligned}\mathcal{B}^*\mathcal{B}\phi_B &= \sum_p (\alpha - p - 1)(\beta_1 + \beta_2 - p + 2)b_p\phi_{\alpha-p-1,\beta_1+1,\beta_2-p,\gamma+p} + \\ &\quad - \sum_p (\alpha - p - 1)(\beta_2 - p)b_p\phi_{\alpha-p-2,\beta_1+1,\beta_2-p-1,\gamma+p+1},\end{aligned}$$

where we observe that the last terms of the second sums cancel (for $p = \beta_2$, or for $p = \alpha$, and $b_\alpha = 0$). Then, since $\Phi = (\phi_{\alpha,\beta_1,\beta_2,\gamma}, 0)$, $\Phi = (0, \phi_{\alpha-1,\beta_1+1,\beta_2,\gamma})$, belong to $\mathcal{E}'_{\alpha,\beta_1,\beta_2,\gamma}$ respectively for $\alpha \geq 0$ and $\alpha \geq 1$ we have the splitting of $\mathcal{H}'_{AB}$ into the non direct sum

$$\mathcal{H}'_{AB} = \sum_{\alpha+\beta_1+\beta_2+2\gamma=k} \mathcal{E}'_{\alpha,\beta_1,\beta_2,\gamma}.$$

Hence, the spectrum $\mathrm{spec}(\mathcal{T}|_{\mathcal{H}'_{\mathrm{AB}}})$ *of the operator* $\mathcal{T}$ *in* $\mathcal{H}'_{AB}$ *is given by the union with possibly many overlaps*

$$\mathrm{spec}(\mathcal{T}|_{\mathcal{H}'_{\mathrm{AB}}}) = \bigcup_{\alpha+\beta_1+\beta_2+2\gamma=\mathrm{k}} \mathrm{spec}(\mathrm{T}|_{\mathcal{E}'_{\alpha,\beta_1,\beta_2,\gamma}}).$$

Step 4.2.2 Spectrum of $\mathcal{T}$ in $\mathcal{E}'_{\alpha,\beta_1,\beta_2,\gamma}$. Denoting

$$\lambda' = \lambda - \omega^2(\alpha + \beta_1 - \beta_2 - 1)^2,$$

the spectral equation $\mathcal{T}\Phi = \lambda\Phi$, for $\phi \in \mathcal{E}'_{\alpha,\beta_1,\beta_2,\gamma}$ can be written as a hierarchy of systems of equations $(45)_p$ where for $p = 0$ we have

$$\begin{aligned} -\lambda' a_0 + (\alpha a_0 - b_0)(\beta_1 + \beta_2 + 1) &= 0 \\ -\lambda' b_0 + b_0 - \alpha a_0 + (\alpha - 1)(\beta_1 + \beta_2 + 2)b_0 &= 0 \end{aligned} \qquad (45)_0$$

and for $1 \leq p \leq \min\{\alpha, \beta_2\}$

$$\begin{aligned} &-\lambda' a_p + \{(\alpha - p)a_p - b_p\}(\beta_1 + \beta_2 - p + 1) \\ &\qquad\qquad - \{(\alpha - p + 1)a_{p-1} - b_{p-1}\}(\beta_2 - p + 1) = 0, \\ &-\lambda' b_p + b_p - (\alpha - p)a_p + (\alpha - p - 1)(\beta_1 + \beta_2 - p + 2)b_p \\ &\qquad\qquad - (\alpha - p)(\beta_2 - p + 1)b_{p-1} = 0. \end{aligned} \qquad (45)_p$$

In the case $\beta_2 \leq \alpha - 1$,the last system is obtained for $p = \beta_2$ and it reads

$$\begin{aligned} -\lambda' a_{\beta_2} + \{(\alpha - \beta_2)a_{\beta_2} - b_{\beta_2}\}(\beta_1 + 1) - \{(\alpha - \beta_2 + 1)a_{\beta_2-1} - b_{\beta_2-1}\} &= 0, \\ -\lambda' b_{\beta_2} + b_{\beta_2} - (\alpha - \beta_2)a_{\beta_2} + (\alpha - \beta_2 - 1)(\beta_1 + 2)b_{\beta_2} - (\alpha - \beta_2)b_{\beta_2-1} &= 0, \end{aligned}$$

whereas in the case $\alpha \leq \beta_2$ the last system is obtained for $p = \alpha$, imposing $b_\alpha = 0$ and it reads

$$\begin{aligned} -\lambda' a_\alpha - (a_{\alpha-1} - b_{\alpha-1})(\beta_2 - \alpha + 1) &= 0 \\ 0 &= 0. \end{aligned}$$

For $|a_0| + |b_0| \neq 0$, the system $(45)_0$ gives two eigenvalues

$$\begin{array}{ll} \lambda'_1 = (\alpha - 1)(\beta_1 + \beta_2 + 1), & a_0 = b_0 = 1, \\ \lambda'_2 = \alpha(\beta_1 + \beta_2 + 2), & a_0 = \beta_1 + \beta_2 + 1, \quad b_0 = -\alpha. \end{array}$$

Let us show that for $\alpha \geq 2$ (which gives non zero eigenvalues λ') one can solve the above system in (a_p, b_p). The determinant of the system $(45)_p$ is given by

$$\begin{aligned} D'_p &= \{-\lambda' + (\alpha - p)(\beta_1 + \beta_2 - p + 1)\}\{-\lambda' + 1 + (\alpha - p - 1)(\beta_1 + \beta_2 - p + 2)\} + \\ &\quad -(\alpha - p)(\beta_1 + \beta_2 - p + 1) \end{aligned}$$

where

$$\{-\lambda' + \alpha(\beta_1 + \beta_2 + 1)\}\{-\lambda' + 1 + (\alpha - 1)(\beta_1 + \beta_2 + 2)\} - \alpha(\beta_1 + \beta_2 + 1) = 0.$$

Hence

$$\begin{aligned} D'_p &= p(\beta_1 + \beta_2 + \alpha + 1 - p)\{2\lambda' - \alpha(\beta_1 + \beta_2 + 1) + \\ &\quad -(\alpha - 1)(\beta_1 + \beta_2 + 2) + p(\beta_1 + \beta_2 + \alpha + 1 - p)\}. \end{aligned}$$

The first factor $(\beta_1 + \beta_2 + \alpha + 1 - p)$ is > 0 in all cases. For $\lambda'_1 = (\alpha - 1)(\beta_1 + \beta_2 + 1)$, the second factor is

$$(p - 1)(\beta_1 + \beta_2 + \alpha - p),$$

while for $\lambda'_2 = \alpha(\beta_1 + \beta_2 + 2)$, the second factor is

$$\beta_1 + \beta_2 + \alpha + 2 + p(\beta_1 + \beta_2 + \alpha + 1 - p) > 0.$$

Finally, the determinant is different from 0 except for $p = 1$ when $\lambda' = \lambda'_1$ and when $\alpha = \beta_1 = \beta_2 = p = 0$. It then remains to study these cases and the case when $\lambda' = 0$, $\alpha \leq \beta_2$ with $\alpha = 0$, or 1 (for computing a_α).

The case $\alpha = 0$ corresponds to an eigenvector of the form $(p = 0)$

$$\begin{aligned} \phi_A &= \phi_{0,\beta_1,\beta_2,\gamma} \\ \phi_B &= 0. \end{aligned}$$

Indeed this is an eigenvector of $\mathcal{T}$ belonging to $\lambda = \omega^2(\beta_1 - \beta_2 - 1)^2$ i.e. $\lambda' = 0$.

We can also check that the case $\alpha = 1$ corresponds to an eigenvector of $\mathcal{T}$ of the form

$$\begin{aligned} \phi_A &= \phi_{1,\beta_1,\beta_2,\gamma} \\ \phi_B &= \phi_{0,\beta_1+1,\beta_2,\gamma} \end{aligned}$$

belonging to the eigenvalue $\lambda = \omega^2(\beta_1 - \beta_2)^2$ i.e. $\lambda' = 0$. The above system reduces to $a_0 = b_0 = 1$ and the equation for a_1 is then $0a_1 = 0$.

The system $(45)_p$ for $p = 1$ and $\lambda' = \lambda'_1$ reads

$$\begin{aligned} -(\alpha - 1)a_1 - (\beta_1 + \beta_2)b_1 &= \beta_2(\alpha a_0 - b_0) \\ -(\alpha - 1)a_1 - (\beta_1 + \beta_2)b_1 &= \beta_2(\alpha - 1)b_0 \end{aligned}$$

which satisfies the compatibility condition, since $a_0 = b_0$. Hence we can indeed compute all coefficients a_p, b_p. This means that the eigenvalue λ'_1 is at least double, which does not give any trouble here (selfadjoint operator).

In conclusion, the above study allows to obtain all possible eigenvalues of $\mathcal{T}$ in $\mathcal{H}'_{AB}$. Since we proved that the only possible values for λ' are nonnegative integers, we get that

$$\min_{\substack{\lambda\in \mathrm{spec}(\mathcal{T}|_{\mathcal{H}'_{\mathrm{AB}}})\\ \lambda\neq 0}} \{\lambda\} \geq \min\{1,\omega^2\}.$$

Step 4.3. Spectrum of $\mathcal{T}$ in $\mathcal{H}''_{AB}$. We now proceed in the same way in the subspace $\mathcal{H}''_{AB}$ and we also split $\mathcal{H}''_{AB}$ into subspaces invariant by $\mathcal{T}$.

Step 4.2.2. Splitting of $\mathcal{H}''_{AB}$. First observe that for $\alpha_1+\alpha_2+\beta+2\gamma = k$, the subspace $\mathcal{E}''_{\alpha_1,\alpha_2,\beta,\gamma}$ of $\mathcal{H}'_{AB}$ gathering the polynomials Φ of the form

$$\begin{aligned} \psi_A &= \sum_p a_p \psi_{\alpha_1-p,\alpha_2,\beta-p,\gamma+p} \\ \psi_B &= \sum_p b_p \psi_{\alpha_1-p,\alpha_2-1,\beta-p+1,\gamma+p} \end{aligned} \tag{46}$$

with $\alpha_2 \geq 1$ and where

$$\begin{array}{ll} \text{for } \beta\leq \alpha_1-1, & 0\leq p\leq \beta+1, \text{ with } a_{\beta+1}=0 \\ \text{for } \alpha_1\leq\beta & 0\leq p\leq \alpha_1, \end{array}$$

is invariant under the operator $\mathcal{T}$. Indeed we have

$$\begin{aligned} \mathcal{B}\psi_A-\psi_B &= \sum_p \{(\alpha_1+\alpha_2-p)a_p - b_p\}\psi_{\alpha_1-p,\alpha_2-1,\beta-p+1,\gamma+p} + \\ &\quad -\sum_p (\alpha_1-p)a_p\psi_{\alpha_1-p-1,\alpha_2-1,\beta-p,\gamma+p+1} \end{aligned}$$

$$\begin{aligned} \mathcal{B}^*(\mathcal{B}\psi_A-\psi_B) &= \sum_p \{(\alpha_1+\alpha_2-p)a_p - b_p\}(\beta-p+1)\psi_{\alpha_1-p,\alpha_2,\beta-p,\gamma+p} + \\ &\quad -\sum_p (\alpha_1-p)(\beta-p)a_p\psi_{\alpha_1-p-1,\alpha_2,\beta-p-1,\gamma+p+1} \end{aligned}$$

$$\begin{aligned} \mathcal{B}^*\mathcal{B}\psi_B &= \sum_p (\alpha_1+\alpha_2-p-1)(\beta-p+2)b_p\psi_{\alpha_1-p,\alpha_2-1,\beta-p+1,\gamma+p} + \\ &\quad -\sum_p (\alpha_1-p)(\beta-p+1)b_p\psi_{\alpha_1-p-1,\alpha_2-1,\beta-p,\gamma+p+1}. \end{aligned}$$

Moreover, $\Phi''_{\alpha_1,\alpha_2} = (0,\psi_{\alpha_1,\alpha_2,0,0})$ is an eigenvector of $\mathcal{A}_{L^*}\mathcal{A}_L$ belonging to the eigenvalue $\lambda = \omega^2(\alpha_1-\alpha_2-1)^2+k+1$.

Then, since $\Phi = (\psi_{\alpha_1,\alpha_2,\beta,\gamma},0)$, $\Phi = (\psi_{\alpha_1,\alpha_2-1,\beta+1,\gamma},0)$, and $\Psi = (\phi_{\alpha_1,\alpha_2-1,\beta,\gamma+1},0)$ belong to $\mathcal{E}'_{\alpha,\beta_1,\beta_2,\gamma}$ respectively for $\alpha_1 \geq 0$,$\alpha_1 \geq 0$ and $\alpha_1 \geq 1$ we have the splitting of $\mathcal{H}'_{AB}$ into the non direct sum

$$\mathcal{H}'_{AB} = \sum_{\alpha+\beta_1+\beta_2+2\gamma=k} \mathcal{E}''_{\alpha_1,\alpha_2,\beta,\gamma} + \operatorname*{span}_{\alpha_1+\alpha_2=k}\{\Phi''_{\alpha_1,\alpha_2}\}.$$

Hence, the spectrum $\mathrm{spec}(\mathcal{T}|_{\mathcal{H}'_{\mathrm{AB}}})$ *of the operator $\mathcal{T}$ in $\mathcal{H}'_{AB}$ is given by the union with possibly many overlaps*

$$\mathrm{spec}(\mathcal{T}|_{\mathcal{H}'_{\mathrm{AB}}}) = \bigcup_{\alpha_1+\alpha_2=\mathrm{k}} \left\{\omega^2(\alpha_1-\alpha_2-1)^2+\mathrm{k}+1\right\} \cup \bigcup_{\alpha_1+\alpha_2+\beta+2\gamma=\mathrm{k}} \mathrm{spec}(\mathcal{T}|_{\mathcal{E}''_{\alpha_1,\alpha_2,\beta,\gamma}}).$$

Step 4.2.2 Spectrum of $\mathcal{T}$ in $\mathcal{E}''_{\alpha_1,\alpha_2,\beta,\gamma}$. Denoting

$$\lambda' = \lambda - \omega^2(\alpha_2 - \alpha_1 + \beta + 1)^2,$$

the spectral equation $\mathcal{T}\Phi = \lambda\Phi$, for $\phi \in \mathcal{E}''_{\alpha_1,\alpha_2,\beta,\gamma}$ can be written as a hierarchy of systems of equations $(45)_p$ where for $p = 0$ we have

$$\begin{array}{rcl} 0 &=& -\lambda' a_0 + \{(\alpha_1+\alpha_2)a_0 - b_0\}(\beta+1) \\ 0 &=& -\lambda' b_0 + b_0 - (\alpha_1+\alpha_2)a_0 + (\alpha_1+\alpha_2-1)(\beta+2)b_0 \end{array} \qquad (47)_0$$

and for $1 \leq p \leq \min\{\alpha_1, \beta+1\}$,

$$\begin{array}{rcl} 0 &=& -\lambda' a_p + \{(\alpha_1+\alpha_2-p)a_p - b_p\}(\beta-p+1) \\ && \qquad -(\alpha_1-p+1)(\beta-p+1)a_{p-1} \\ 0 &=& -\lambda' b_p + b_p - (\alpha_1+\alpha_2-p)a_p + (\alpha_1+\alpha_2-p-1)(\beta-p+2)b_p \\ && +(\alpha_1-p+1)\{a_{p-1} - (\beta-p+2)b_{p-1}\} \end{array} \qquad (47)_p$$

In the case $\beta \leq \alpha_1 - 1$, the last equation is obtain for $p = \beta+1$ and it reads (using $a_{\beta+1} = 0$)

$$\begin{array}{rcl} 0 &=& -\lambda' a_\beta + \{(\alpha_1+\alpha_2-\beta)a_\beta - b_\beta\} - (\alpha_1-\beta+1)a_{\beta-1} \\ 0 &=& -\lambda' b_\beta + b_\beta - (\alpha_1+\alpha_2-\beta)a_\beta + (\alpha_1+\alpha_2-\beta-1)2b_\beta + \\ && +(\alpha_1-\beta+1)\{a_{\beta-1} - 2b_{\beta-1}\}, \end{array}$$

which should be completed by (using $a_{\beta+1} = 0$)

$$\begin{array}{rcl} 0 &=& 0 \\ 0 &=& -\lambda' b_{\beta+1} + (\alpha_1+\alpha_2-\beta-1)b_{\beta+1} - (\alpha_1-\beta)(b_\beta - a_\beta), \end{array}$$

and the last equation ($p = \alpha_1$), obtained in the case $\alpha_1 \leq \beta$ is

$$\begin{array}{rcl} 0 &=& -\lambda' a_{\alpha_1} + \{\alpha_2 a_{\alpha_1} - b_{\alpha_1}\}(\beta-\alpha_1+1) - (\beta-\alpha_1+1)a_{\alpha_1-1} \\ 0 &=& -\lambda' b_{\alpha_1} + b_{\alpha_1} - \alpha_2 a_{\alpha_1} + (\alpha_2-1)(\beta-\alpha_1+2)b_{\alpha_1} + \\ && +a_{\alpha_1-1} - (\beta-\alpha_1+2)b_{\alpha_1-1}. \end{array}$$

If $|a_0| + |b_0| \neq 0$, the system ($(47)_0$) gives two eigenvalues

$$\begin{array}{rcl} \lambda'_1 &=& (\beta+1)(\alpha_1+\alpha_2-1) \geq 0 \\ \lambda'_2 &=& (\beta+2)(\alpha_1+\alpha_2) > 0 \end{array}$$

corresponding respectively to the two eigenvectors

$$\begin{array}{rcl} a_0 &=& b_0 = 1, \\ a_0 &=& \beta+1, \quad b_0 = -(\alpha_1+\alpha_2). \end{array}$$

Notice that even in the case when $\lambda'_1 = 0$, one has $\lambda \geq 4\omega^2$. Let us show that we can now determine the components a_p, b_p for $p > 0$. The determinant of the system ($(47)_p$) is given by

$$\begin{array}{rcl} D''_p &=& \{-\lambda' + (\alpha_1+\alpha_2-p)(\beta-p+1)\}\{-\lambda' + 1 + (\alpha_1+\alpha_2-p-1)(\beta-p+2)\} + \\ && -(\beta-p+1)(\alpha_1+\alpha_2-p) \end{array}$$

where

$$0=\{-\lambda'+(\alpha_1+\alpha_2)(\beta+1)\}\{-\lambda'+1+(\alpha_1+\alpha_2-1)(\beta+2)\}-(\beta+1)(\alpha_1+\alpha_2).$$

Hence

$$\begin{aligned} D_p'' &= p(\beta+\alpha_1+\alpha_2+1-p)\{2\lambda'-(\alpha_1+\alpha_2)(\beta+1)+ \\ & -(\alpha_1+\alpha_2-1)(\beta+2)+p(\beta+\alpha_1+\alpha_2+1-p)\}. \end{aligned}$$

The first factor is positive for $p\le\beta+1\le\alpha_1$, and for $p\le\alpha_1\le\beta$. For $\lambda'=\lambda_1'$, the second factor is

$$(p-1)(\alpha_1+\alpha_2+\beta-p)$$

while for λ_2' the second factor is

$$\beta+\alpha_1+\alpha_2+2+p(\beta+\alpha_1+\alpha_2+1-p)>0.$$

The determinants are then different from 0 except for $p=1$, with $\lambda'=\lambda_1'$, and in the special cases when $\alpha_1=\alpha_2=\beta=p=0$. This last case was already seen since it corresponds to the eigenvector.

$$\phi_A=\phi_{000\gamma},\quad \phi_B=0.$$

The equation $((47)_p)$ for $p=1$ and $\lambda'=\lambda_1'$ leads to

$$\begin{aligned} -(\alpha_1+\alpha_2-1)a_1-\beta b_1 &= \alpha_1\beta a_0 \\ -(\alpha_1+\alpha_2-1)a_1-\beta b_1 &= -\alpha_1\{a_0-(\beta+1)b_0\} \end{aligned}$$

where the compatibility condition is satisfied due to $a_0=b_0$. This means that λ_1' is at least a double eigenvalue and that we can compute all coefficients a_p,b_p.

In conclusion, the above study allows to obtain all possible eigenvalues of $\mathcal{T}$ in $\mathcal{H}''_{AB}$. Since we proved that the only possible values for λ' are nonnegative integers, we get that

$$\min_{\substack{\lambda\in\mathrm{spec}(\mathcal{T}|_{\mathcal{H}''_{\mathrm{AB}}})\\ \lambda\neq 0}}\{\lambda\}\ge\min\{1,\omega^2\}.$$

Finally since $\mathcal{H}_{AB}=\mathcal{H}'_{AB}\oplus\mathcal{H}''_{AB}$ we get that

$$\min_{\substack{\lambda\in\mathrm{spec}(\mathcal{T}|_{\mathcal{H}_{\mathrm{AB}}})\\ \lambda\neq 0}}\{\lambda\}\ge\min\{1,\omega^2\}.$$

The same estimates holds in $\mathcal{H}_{\widetilde{\widetilde{AB}}}$. Hence we can conclude that $\min\limits_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\}\ge\min\{1,\omega^2\}$ and thus

$$a_k(L):=|||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2\le\max\{1,\omega^{-1}\}.$$

□

3 Exponential estimates for perturbed vector fields

Pour finir je donnerai ici les résultats pour le cas des champ de vecteurs avec paramètres.

A Properties of the normalized euclidian norm

A.1 Comparison of the euclidian and the sup norm

We begin with two technical lemmas which will be used several times

Lemma A.1 *Let k,m be two positive integers and $\{u_j\}_{1\le j\le m}$ be m complex numbers. Then*

$$\frac{(u_1+\cdots+u_m)^k}{k!}=\sum_{|\alpha|=k}\frac{u_1^{\alpha_1}}{\alpha_1!}\cdots\frac{u_m^{\alpha_m}}{\alpha_m!}$$

Proof. We proceed by induction. For $m=1$ this is trivial and for $m=2$ this is true because of the binomial formula. Assume now that it is true for $m\ge 2$, then

$$\begin{aligned}\sum_{|\alpha|=k}\frac{u_1^{\alpha_1}}{\alpha_1!}\cdots\frac{u_{m+1}^{\alpha_{m+1}}}{\alpha_{m+1}!} &= \sum_{\alpha_{m+1}=0}^{k}\frac{u_{m+1}^{\alpha_{m+1}}}{\alpha_{m+1}!}\sum_{\alpha_1+\cdots+\alpha_m=k-\alpha_{m+1}}\frac{u_1^{\alpha_1}}{\alpha_1!}\cdots\frac{u_m^{\alpha_m}}{\alpha_m!}\\ &= \sum_{\alpha_{m+1}=0}^{k}\frac{u_{m+1}^{\alpha_{m+1}}}{\alpha_{m+1}!}\,\frac{(u_1+\cdots+u_m)^{k-\alpha_{m+1}}}{(k-\alpha_{m+1})!}\\ &= \frac{(u_1+\cdots+u_{m+1})^k}{k!}.\end{aligned}$$

□

Lemma A.2 *Let k,m be two positive integers and*

$$\begin{aligned}\mathcal{E}^1_{k,m}&=\{\beta=(\beta_1,\cdots,\beta_m)\in\mathbb{N}^m,\beta_j\ge 1,|\beta|=k\},\\ \mathcal{E}^0_{k,m}&=\{\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbb{N}^m,\alpha_j\ge 0,|\alpha|=k\}.\end{aligned}$$

Then, the cardinals $d^j_{k,m}$ of $\mathcal{E}^j_{k,m}$, $j=0,1$, are given by

$$d^1_{k,m}=C^{m-1}_{k-1},\qquad d^0_{k,m}=C^{m-1}_{k+m-1}.$$

where $C^r_n=\dfrac{n!}{r!(n-r)!}$.

Proof. The cardinal of $d^1_{k,m}$ is equal to the number of ways for placing $(m-1)$ distinct separators among $k-1$ possible locations, the order of the separators being meaningless. For instance, the cardinal of $d^1_{k,3}$, is equal to the number of ways for placing 2 distinct separators among $k-1$ possible locations, the order of the separators being meaningless.

$$\overbrace{[\underbrace{\cdot\,|\,\cdot\,|\,\cdot}_{\alpha_1}|\underbrace{\cdot\,|\,\cdot}_{\alpha_2}|\underbrace{\cdot\,|\,\cdot\,|\,\cdot\,|\,\cdot}_{\alpha_3}]}^{k}$$

Hence, $d^1_{k,3}=C^2_{k-1}$ and more generally, $d^1_{k,m}=C^{m-1}_{k-1}$.

Finally, the map $\mathcal{E}^0_{k,m} \to \mathcal{E}^1_{k+m,m} : (\alpha_1, \cdots, \alpha_m) \mapsto (\beta_1 := \alpha_1 + 1, \cdots, \beta_m := \alpha_m + 1)$ is one to one. Hence

$$d^0_{k,m} = d^1_{m+k} = C^{m-1}_{m+k-1}.$$

$\square$

Lemma A.3 *For every $\Phi \in \mathcal{H}_k$,* $\quad |\Phi|_{0,k} \leq |\Phi|_{2,k} = \frac{1}{\sqrt{k!}} |\Phi|_2$.

Proof. For $\Phi \in \mathcal{H}_k$ with $\Phi = \sum\limits_{\substack{1\leq j\leq m\\ |\alpha|=n}} \Phi_{j,\alpha} Y_1^{\alpha_1} \cdots Y_m^{\alpha_m}\, c_j$ where $\{c_j\}_{1\leq j\leq m}$ is the canonical basis of $\mathbb{R}^m$ we have

$$|\Phi|_{2,k} = \frac{1}{\sqrt{k!}} \sqrt{\sum_{\substack{1\leq j\leq m\\ |\alpha|=k}} |\Phi_{j,\alpha}|^2 \alpha_1!.\cdots.\alpha_m!}$$

and

$$\begin{aligned} \frac{\|\Phi(Y)\|^2}{\|Y\|^{2k}} &= \sum_{j=1}^m \Big| \sum_{|\alpha|=k} \Phi_{j,\alpha} \frac{Y_1^{\alpha_1}}{\|Y\|^{\alpha_1}} \cdots \frac{Y_m^{\alpha_m}}{\|Y\|^{\alpha_m}} \Big|^2 \\ &\leq \sum_{j=1}^m \Big(\sum_{|\alpha|=k} |\Phi_{j,\alpha}|^2\ \alpha_1! \cdots \alpha_m! \Big) \Big(\sum_{|\alpha|=k} \frac{Y_1^{2\alpha_1}}{\alpha_1! \|Y\|^{2\alpha_1}} \cdots \frac{Y_m^{2\alpha_m}}{\alpha_m! \|Y\|^{2\alpha_m}} \Big) \end{aligned}$$

by the Cauchy Schwarz formula. Then using Lemma A.1 we get

$$\sum_{|\alpha|=k} \frac{Y_1^{2\alpha_1}}{\alpha_1! \|Y\|^{2\alpha_1}} \cdots \frac{Y_m^{2\alpha_m}}{\alpha_m! \|Y\|^{2\alpha_m}} = \frac{1}{k!} \left(\frac{Y_1^2}{\|Y\|^2} + \cdots + \frac{Y_m^2}{\|Y\|^2} \right)^k = \frac{1}{k!}.$$

Hence,

$$|\Phi|_{0,k} = \sup_{Y\in\mathbb{C}^k\setminus\{0\}} \frac{\|\Phi(Y)\|}{\|Y\|^k} \leq \sqrt{\frac{1}{k!} \sum_{j=1}^m \Big(\sum_{|\alpha|=k} |\Phi_{j,\alpha}|^2\ \alpha_1! \cdots \alpha_m! \Big)} = \frac{1}{\sqrt{k!}} |\Phi|_2 = |\phi|_{2,k}.$$

$\square$

We now prove a Parseval like formula :

Lemma A.4 *For every $\Phi \in \mathcal{H}_k$,*

$$|\Phi|_2^2 = \frac{1}{(2\pi)^m} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_m \int_0^{+\infty} dr_1 \cdots \int_0^{+\infty} dr_m\ \|\Phi(\sqrt{r_1}\mathrm{e}^{i\theta_1}, \cdots, \sqrt{r_m}\mathrm{e}^{i\theta_m})\|^2 \mathrm{e}^{-r_1} \cdots \mathrm{e}^{-r_m}$$

Proof. We have

$$\|\Phi(\sqrt{r_1}\mathrm{e}^{i\theta_1}, \cdots, \sqrt{r_m}\mathrm{e}^{i\theta_m})\|^2 = \sum_{j=1}^m \sum_{\substack{|\alpha|=k\\ |\beta|=k}} \Phi_{j,\alpha} \overline{\Phi_{j,\beta}}\, r_1^{\frac{\alpha_1+\beta_1}{2}} \cdots r_m^{\frac{\alpha_m+\beta_m}{2}}\ \mathrm{e}^{\mathrm{i}\theta_1(\alpha_1-\beta_1)} \cdots \mathrm{e}^{\mathrm{i}\theta_m(\alpha_m-\beta_m)}.$$

Hence,

$$
\begin{aligned}
&\frac{1}{(2\pi)^m}\int_0^{2\pi}\!d\theta_1\cdots\int_0^{2\pi}\!d\theta_m\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\|\Phi(\sqrt{r_1}\mathrm{e}^{i\theta_1},\cdots,\sqrt{r_m}\mathrm{e}^{i\theta_m})\|^2\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m}\\
&=\sum_{j=1}^m\sum_{|\alpha|=k}|\Phi_{j,\alpha}|^2\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\ r_1^{\alpha_1}\cdots r_m^{\alpha_m}\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m}\\
&=\sum_{j=1}^m\sum_{|\alpha|=k}|\Phi_{j,\alpha}|^2\alpha_1!\cdots\alpha_m!=|\Phi|_2^2.
\end{aligned}
$$

□

Finally, we ready to prove the opposite comparison of the two norms in $\mathcal{H}_k$.

Lemma A.5 *For every $\Phi\in\mathcal{H}_k$,* $|\Phi|_{2,k}\le\sqrt{C_{k+m-1}^{m-1}}\ |\Phi|_{0,k}$.

Proof. Using the Lemmas A.1, A.4 we get

$$
\begin{aligned}
|\Phi|_{2,k}^2 &\le |\Phi|_{0,k}^2\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\ \frac{(r_1+\cdots+r_m)^k}{k!}\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m},\\
&= |\Phi|_{0,k}^2\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\ \sum_{|\alpha|=k}\frac{r_1^{\alpha_1}}{\alpha_1!}\cdots\frac{r_m^{\alpha_m}}{\alpha_m!}\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m},\\
&= |\Phi|_{0,k}^2\sum_{|\alpha|=k}1,\\
&= |\Phi|_{0,k}^2\ C_{m+k-1}^{m-1},
\end{aligned}
$$

□

A.2 Multiplicativity of the normalized Euclidian Norm

To handle the computations, we need in this subsection more compact notations. For $Y=(Y_1,\cdots,Y_m)\in\mathbb{C}^m$ and $\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbb{N}^m$ let us denote

$$\alpha!=\alpha_1!\cdots\alpha_m!\qquad\text{and}\qquad Y^\alpha=Y_1^{\alpha_1}\cdots Y_m^{\alpha_m}.$$

With these notations, for $\Phi\in\mathcal{H}_n$ with $\Phi(Y)=\sum_{|\alpha|=n}\Phi_\alpha Y^\alpha$ where $\Phi_\alpha\in\mathbb{R}^m$, we have

$$|\Phi|_{2,n}=\frac{1}{\sqrt{n!}}\sqrt{\sum_{|\alpha|=n}\|\Phi_\alpha\|^2\ \alpha!}$$

We start with two technical lemmas which will be used several times.

Lemma A.6 *For $\alpha\in\mathbb{N}^m$ and $n\in\mathbb{N}$ let us denote*

$$\mathfrak{B}_n^\alpha=\frac{n!}{\alpha!}$$

Then for every positive integers q and $\{p_\ell\}_{1\le\ell\le q}$ and every $\gamma\in\mathbb{N}^m$ with $|\gamma|=p_1+\cdots+p_q$, we have

$$\mathfrak{B}_{p_1+\cdots+p_q}^\gamma=\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}\mathfrak{B}_{p_1}^{\alpha^{(1)}}\cdots\mathfrak{B}_{p_q}^{\alpha^{(q)}}.$$

Proof. Using Lemma A.1 we get that for every $u = (u_1, \cdots, u_m) \in \mathbb{C}^m$,

$$\begin{aligned}(u_1 + \cdots + u_m)^{p_1+\cdots+p_q} &= \sum_{|\gamma|=p_1+\cdots+p_q} \mathfrak{B}^{\gamma}_{p_1+\cdots+p_q}\ u^{\gamma},\\ &= (u_1 + \cdots + u_m)^{p_1} \cdots (u_1 + \cdots + u_m)^{p_q},\\ &= \sum_{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell} \mathfrak{B}^{\alpha^{(1)}}_{p_1} \cdots \mathfrak{B}^{\alpha^{(q)}}_{p_q}\ u^{\alpha^{(1)}+\cdots+\alpha^{(q)}}.\end{aligned}$$

Identifying the powers of u we get the desired result. $\square$

Lemma A.7 *Let $k > 0, p \geq 0$ be two integers. Then for every $\gamma \in \mathbb{N}^m$ with $|\gamma| = n$ with $n := k - 1 + p$*

$$(k^2 + (m-1)k)\ \mathfrak{B}^{\gamma}_n = \sum_{j=1}^{m} \sum_{\substack{|\alpha|=k, |\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} (\alpha_j)^2\ \mathfrak{B}^{\alpha}_k\ \mathfrak{B}^{\beta}_p.$$

where $\sigma_j = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{N}^m$ with the coefficient 1 placed at the j-th position.

Proof. Observe that for every $u = (u_1, \cdots, u_m) \in \mathbb{C}^m$,

$$(u_1 + \cdots + u_m)^p \sum_{j=1}^{m} \left(u_j \frac{\partial^2}{\partial u_j^2} + \frac{\partial}{\partial u_j} \right) \left((u_1 + \cdots + u_m)^k \right) = (k^2 + (m-1)k)(u_1 + \cdots + u_m)^n.$$

Hence, since $\left(u_j \frac{\partial^2}{\partial u_j^2} + \frac{\partial}{\partial u_j} \right) u^{\alpha} = (\alpha^j)^2 u^{\alpha-\sigma_j}$, we get

$$(k^2 + (m-1)k) \sum_{|\gamma|=n} \mathfrak{B}^{\gamma}_q\ u^{\gamma} = \sum_{j=1}^{m} \sum_{\substack{|\alpha|=k, \\ |\beta|=p}} (\alpha_j)^2\ \mathfrak{B}^{\alpha}_k\ \mathfrak{B}^{\beta}_p\ u^{\alpha+\beta-\sigma_j}$$

Identifying the powers of u we immediately get the desired result. $\square$

We are now ready to prove the multiplicativity of the normalized euclidian norm in $\mathcal{H}_n$.

Lemma A.8 *Let q and $\{p_\ell\}_{1\leq \ell \leq q}$ be positive integers and let $R_q \in \mathcal{L}_q(\mathbb{R}^m)$ be q-linear. Then for every $\Phi_{p_\ell} \in \mathcal{H}_{p_\ell}$, $1 \leq \ell \leq q$, the polynomial $R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]$ lies in $\mathcal{H}_n$ with $n = p_1 + \cdots + p_q$ and*

$$\left| R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}] \right|_{2,n} \leq |||R_q|||_{\mathcal{L}_q(\mathbb{R}^m)} \left|\Phi_{p_1}\right|_{2,p_1} \cdots \left|\Phi_{p_q}\right|_{2,p_q}.$$

Proof. For $1 \leq \ell \leq q$, let us denote

$$\Phi_{p_\ell}(Y) = \sum_{|\alpha|=p_\ell} \Phi^{(p_\ell)}_{\alpha}\ Y^{\alpha}$$

Since R_q is q-linear we get

$$\begin{aligned}R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}] &= \sum_{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell} Y^{\alpha^{(1)}+\cdots+\alpha^{(q)}}\ R_q[\Phi^{(p_1)}_{\alpha^{(1)}}, \cdots, \Phi^{(p_q)}_{\alpha^{(q)}}],\\ &= \sum_{|\gamma|=n} Y^{\gamma} \sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} R_q[\Phi^{(p_1)}_{\alpha^{(1)}}, \cdots, \Phi^{(p_q)}_{\alpha^{(q)}}].\end{aligned}$$

Hence,

$$
\begin{aligned}
\left|R_q[\Phi_{p_1},\cdots,\Phi_{p_q}]\right|^2_{2,n} &= \frac{1}{n!}\sum_{|\gamma|=n}\gamma!\,\Big\|\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} R_q[\Phi^{(p_1)}_{\alpha^{(1)}},\cdots,\Phi^{(p_q)}_{\alpha^{(q)}}]\ \Big\|^2,\\
&\le \sum_{|\gamma|=n}\frac{1}{\mathfrak{B}^\gamma_n}\Big(\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}|||R_q|||_{\mathcal{L}_q(\mathbb{R}^m)}\|\Phi^{(p_1)}_{\alpha^{(1)}}\|\cdots\|\Phi^{(p_q)}_{\alpha^{(q)}}\|\ \Big)^2,\\
&\le |||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}\sum_{|\gamma|=n}\Big[\frac{1}{\mathfrak{B}^\gamma_n}\Big(\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}(\alpha^{(1)}!\|\Phi^{(p_1)}_{\alpha^{(1)}}\|^2)\cdots(\alpha^{(q)}!\|\Phi^{(p_q)}_{\alpha^{(q)}}\|^2)\ \Big)\\
&\qquad\qquad\times\Big(\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}\frac{1}{\alpha^{(1)}!}\cdots\frac{1}{\alpha^{(q)}!}\ \Big)\Big],
\end{aligned}
$$

by the Cauchy-Schwarz formula. Then since Lemma A.6 ensures that

$$
\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}\frac{1}{\alpha^{(1)}!}\cdots\frac{1}{\alpha^{(q)}!}=\frac{1}{p_1!\cdots p_q!}\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}\mathfrak{B}^{\alpha^{(1)}}_{p_1}\cdots\mathfrak{B}^{\alpha^{(q)}}_{p_q}=\frac{1}{p_1!\cdots p_q!}\mathfrak{B}^\gamma_n,
$$

we obtain

$$
\begin{aligned}
\left|R_q[\Phi_{p_1},\cdots,\Phi_{p_q}]\right|^2_{2,n} &\le \frac{|||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}}{p_1!\cdots p_q!}\sum_{|\gamma|=n}\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}(\alpha^{(1)}!\|\Phi^{(p_1)}_{\alpha^{(1)}}\|^2)\cdots(\alpha^{(q)}!\|\Phi^{(p_q)}_{\alpha^{(q)}}\|^2),\\
&= \frac{|||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}}{p_1!\cdots p_q!}\sum_{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell}(\alpha^{(1)}!\|\Phi^{(p_1)}_{\alpha^{(1)}}\|^2)\cdots(\alpha^{(q)}!\|\Phi^{(p_q)}_{\alpha^{(q)}}\|^2),\\
&= |||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}\prod_{\ell=1}^q\Big(\frac{1}{p_\ell!}\sum_{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell}\alpha^{(\ell)}!\|\Phi^{(p_\ell)}_{\alpha^{(\ell)}}\|^2\Big),\\
&= |||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}\left|\Phi_{p_1}\right|^2_{2,p_1}\cdots\left|\Phi_{p_q}\right|^2_{2,p_q}.
\end{aligned}
$$

□

Lemma A.9 *Let $k>0$, $p\ge 0$ be two integers and let Φ_k, N_p lie respectively in $\mathcal{H}_k$ and $\mathcal{H}_p$. Then $D\Phi_k.N_p$ lies in $\mathcal{H}_n$ with $n=k-1+p$ and*

$$
\left|D\Phi_k.N_p\right|_{2,n}\le\sqrt{k^2+(m-1)k}\ \left|\Phi_p\right|_{2,k}\left|N_p\right|_{2,p}
$$

Proof. Let us denote

$$
\Phi_k(Y)=\sum_{|\alpha|=k}Y^\alpha\Phi_\alpha,\qquad N_p(Y)=\sum_{|\beta|=p}Y^\beta N_\beta
$$

where $\Phi_\alpha, N_\beta\in\mathbb{C}^m$, and $N_\beta=(N_{\beta,1},\cdots,N_{\beta,m})$. Then,

$$
D\Phi_k.N_p=\sum_{j=1}^m\sum_{\substack{|\alpha|=k\\|\beta|=p}}\alpha_j\ Y^{\alpha-\sigma_j+\beta}N_{\beta,j}\Phi_\alpha=\sum_{|\gamma|=n}Y^\gamma\sum_{j=1}^m\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}\alpha_jN_{\beta,j}\Phi_\alpha.
$$

where $\sigma_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ with the coefficient 1 placed at the j-th position. Hence,

$$\begin{aligned}
|D\Phi_k.N_p|^2_{2,n} &\leq \sum_{|\gamma|=n} \frac{1}{\mathfrak{B}_n^\gamma} \Big(\sum_{j=1}^m \sum_{\substack{|\alpha|=k,|\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} \alpha_j |N_{\beta,j}| \|\Phi_\alpha\| \Big)^2 \\
&\leq \sum_{|\gamma|=n} \frac{1}{\mathfrak{B}_n^\gamma} \Big[\Big(\sum_{j=1}^m \sum_{\substack{|\alpha|=k,|\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} \alpha!\beta! |N_{\beta,j}|^2 \|\Phi_\alpha\|^2 \Big), \\
&\qquad\qquad \times \Big(\sum_{j=1}^m \sum_{\substack{|\alpha|=k,|\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} (\alpha_j)^2 \frac{1}{\alpha!} \frac{1}{\beta!} \Big) \Big]
\end{aligned}$$

by the Cauchy-Schwarz formula. Then, since Lemma A.7 ensures that

$$\sum_{j=1}^m \sum_{\substack{|\alpha|=k,|\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} \alpha_j^2 \frac{1}{\alpha!} \frac{1}{\beta!} = \frac{1}{k!p!} \sum_{j=1}^m \sum_{\substack{|\alpha|=k,|\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} \alpha_j^2 \, \mathfrak{B}_k^\alpha \mathfrak{B}_p^\beta = \frac{1}{k!p!} (k^2 + (m-1)k) \mathfrak{B}_n^\gamma$$

we finally obtain

$$\begin{aligned}
|D\Phi_k.N_p|^2_{2,n} &\leq \frac{1}{k!p!} (k^2 + (m-1)k) \sum_{|\gamma|=n} \sum_{j=1}^m \sum_{\substack{|\alpha|=k,|\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} \alpha!\beta! |N_{\beta,j}|^2 \|\Phi_\alpha\|^2 \\
&= \frac{1}{k!p!} (k^2 + (m-1)k) \sum_{\substack{|\alpha|=k, \\ |\beta|=p}} \sum_{j=1}^m \alpha!\beta! \|\Phi_\alpha\|^2 |N_{\beta,j}|^2 \\
&= \frac{1}{k!p!} (k^2 + (m-1)k) \sum_{\substack{|\alpha|=k, \\ |\beta|=p}} \alpha!\beta! \|\Phi_\alpha\|^2 \|N_\beta\|^2 \\
&= (k^2 + (m-1)k) \;\; |\Phi_k|^2_{2,k} \, |N_p|^2_{2,p}
\end{aligned}$$

□

A.3 Invariance of the euclidian norm under unitary linear change of coordinates

Lemma A.10 *Let Q be a unitary linear map in $\mathbb{R}^m$ or $\mathbb{C}^m$ and denote $\mathcal{T}_Q : \mathcal{H} \to \mathcal{H}, \Phi \mapsto Q^{-1} \circ \Phi \circ Q$. Then $\mathcal{T}_Q$ is a unitary linear operator in H, i.e. for every $\Phi \in \mathcal{H}$,*

$$\left| \mathcal{T}_Q \Phi \right|_2 = |\Phi|_2 .$$

Proof. Using lemma A.4 we get that

$$\left| \mathcal{T}_Q \Phi \right|_2^2 = \frac{1}{(2\pi)^m} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_m \int_0^{+\infty} dr_1 \cdots \int_0^{+\infty} dr_m \, \|\Phi \circ Q(\sqrt{r_1} \mathrm{e}^{i\theta_1}, \cdots, \sqrt{r_m} \mathrm{e}^{i\theta_m})\|^2 \mathrm{e}^{-r_1} \cdots \mathrm{e}^{-r_m}.$$

Then performing the change of coordinates $(r_1, \cdots, r_m, \theta_1, \cdots, \theta_m) \mapsto (r'_1, \cdots, r'_m, \theta'_1, \cdots, \theta'_m)$ with

$$(\sqrt{r'_1} \mathrm{e}^{i\theta'_1}, \cdots, \sqrt{r'_m} \mathrm{e}^{i\theta'_m}) = Q(\sqrt{r_1} \mathrm{e}^{i\theta_1}, \cdots, \sqrt{r_m} \mathrm{e}^{i\theta_m})$$

the Jacobian of which is equal to 1 and observing that

$$r_1' + \cdots + r_m' = \|Q(\sqrt{r_1}e^{i\theta_1}, \cdots, \sqrt{r_m}e^{i\theta_m})\| = r_1 + \cdots + r_m$$

we get the desired result. □

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